# Supplementary Material to "An Econometric Model of Network Formation with an Application to Board Interlocks Between Firms"

Cristina Gualdani*

July 2019

## E Inference

After having observed that Artstein's inequalities in expression (6) of the paper are moment inequalities, inference on $\Theta^{\star\star}$ or $\Theta^{o}$ can be performed choosing among several available techniques: if $\mathcal{X}$ is finite e.g., Chernozhukov, Hong, and Tamer (2007), Romano and Shaikh (2008; 2010), Rosen (2008), Andrews and Soares (2010), Romano, Shaikh, and Wolf (2014), Pakes, et al. (2015), Bugni, et al. (2016), Bugni, Canay, and Shi (2017), Kaido, Molinari, and Stoye (2019); if $\mathcal{X}$ is not finite e.g., Andrews and Shi (2013), Chernozhukov, Lee, and Rosen (2013), Andrews and Shi (2017).

Given $\alpha \in (0,1)$, in what follows we illustrate how to construct a $(1-\alpha)$ confidence region for each $\theta \in \Theta^{o}$, by using the generalised moment selection procedure in Andrews and Soares (2010) (hereafter, AS). Equivalent steps can be applied to construct a confidence region for each $\theta \in \Theta^{\star\star}$. We assume that $\mathcal{X}$ is finite and that the observed networks are i.i.d. For simplicity of exposition and without loss of generality, we also assume that the observed networks have the same number, $n$, of players. Moreover, we focus on scenarios where the players' identities vary across the observed networks and the players have no roles.[1]

First, it is convenient to express $\Theta^{o}$ as

$$\Theta^{o} \equiv \Big\{\theta \in \Theta | H^{l}_{g_{\cdot j},\boldsymbol{x}}(\theta) \leq \mathbb{P}(G_{\cdot j} = g_{\cdot j} | \boldsymbol{X} = \boldsymbol{x}) \leq H^{u}_{g_{\cdot j},\boldsymbol{x}}(\theta)\ \forall g_{\cdot j} \in \{0,1\}^{n-1},\ \forall j \in \mathcal{N},\ \forall \boldsymbol{x} \in \mathcal{X}^{n}\Big\},$$

where

$$H^{l}_{g_{\cdot j},\boldsymbol{x}}(\theta) \equiv \int_{e_{\cdot j} \in \mathbb{R}^{n-1} \text{ s.t. } \mathcal{S}_{j,\theta_u}(\boldsymbol{x}, e_{\cdot j}) = \{g_{\cdot j}\}} dF_j(e_{\cdot j}; \theta_\epsilon),$$

*I gratefully acknowledge financial support from the European Research Council under the European Community's Seventh Framework Program FP7/2007-2013 grant agreement N° 295298. The views expressed in this paper are those of the author and do not involve the responsibility of the Bank of Italy. All errors are mine.

*Email: cristina.gualdani@tse-fr.eu, Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France.

[1] See Section 3.3 in the paper for more details.

and

$$H^u_{g_{\cdot j},\boldsymbol{x}}(\theta) \equiv \int_{e_{\cdot j}\in\mathbb{R}^{n-1}\text{ s.t. } g_{\cdot j}\in\mathcal{S}_{j,\theta_u}(\boldsymbol{x},e_{\cdot j})} dF_j(e_{\cdot j};\theta_\epsilon).$$

This is equivalent to

$$\Theta^o = \Big\{\theta\in\Theta|\mathbb{P}(G_{\cdot j}=g_{\cdot j},\boldsymbol{X}=\boldsymbol{x})-\tilde{H}^l_{g_{\cdot j},\boldsymbol{x}}(\theta)\geq 0,$$
$$\tilde{H}^u_{g_{\cdot j},\boldsymbol{x}}(\theta)-\mathbb{P}(G_{\cdot j}=g_{\cdot j},\boldsymbol{X}=\boldsymbol{x})\geq 0\ \forall g_{\cdot j}\in\{0,1\}^{n-1},\ \forall j\in\mathcal{N},\ \forall\boldsymbol{x}\in\mathfrak{X}^n\Big\},$$

where

$$\tilde{H}^l_{g_{\cdot j},\boldsymbol{x}}(\theta)\equiv H^l_{g_{\cdot j},\boldsymbol{x}}(\theta)\mathbb{P}(\boldsymbol{X}=\boldsymbol{x}),$$

and

$$\tilde{H}^u_{g_{\cdot j},\boldsymbol{x}}(\theta)\equiv H^u_{g_{\cdot j},\boldsymbol{x}}(\theta)\mathbb{P}(\boldsymbol{X}=\boldsymbol{x}).$$

### E.1 Preliminary step

A preliminary step is the estimation of $\mathbb{P}(G_{\cdot j}=g_{\cdot j},\boldsymbol{X}=\boldsymbol{x})$, $\tilde{H}^l_{g_{\cdot j},\boldsymbol{x}}(\theta)$, and $\tilde{H}^u_{g_{\cdot j},\boldsymbol{x}}(\theta)$, $\forall g_{\cdot j}\in\{0,1\}^{n-1}$, $\forall j\in\mathcal{N}$, $\forall\boldsymbol{x}\in\mathfrak{X}^n$, and $\forall\theta\in\Theta$.

Estimating $\mathbb{P}(G_{\cdot j}=g_{\cdot j},\boldsymbol{X}=\boldsymbol{x})$, $\tilde{H}^l_{g_{\cdot j},\boldsymbol{x}}(\theta)$, and $\tilde{H}^u_{g_{\cdot j},\boldsymbol{x}}(\theta)$, for example via a frequency estimator, is complicated by the fact that the players' identities vary across the observed networks and the players have no roles. In this case, the labels are assigned arbitrarily to the players within each network. When replacing probabilities with sample analogues, the researcher needs to ensure that the label assignment does not affect estimates. We achieve that by adding sufficient conditions such that the probability distribution of $\boldsymbol{G}$ conditional on $\boldsymbol{X}$ is invariant under permutations of the labels, as suggested by Sheng (2016). This amount to impose joint exchangeability of networks. More formally, $\boldsymbol{G}$ is jointly exchangeable if

$$\mathbb{P}(\boldsymbol{G}\in K|\boldsymbol{X}=\boldsymbol{x})=\mathbb{P}(\boldsymbol{G}\in K^\varphi|\boldsymbol{X}=\boldsymbol{x}^\varphi),$$

for every permutation $\varphi$ of the labels in $\mathcal{N}$, $\forall K\subseteq\mathcal{G}$, and $\forall\boldsymbol{x}\in\mathfrak{X}^n$, where $K^\varphi$ and $\boldsymbol{x}^\varphi$ are obtained by applying $\varphi$ respectively to $K$ and $\boldsymbol{x}$ (Kallenberg, 2005).

The procedure is described below in 6 steps.

**Step 1** This step imposes sufficient conditions for joint exchangeability of networks, as in Sheng (2016).

**Assumption E.1.** (Exchangeability)

a) The finite sequence of random variables $\{\epsilon_{ij}\}_{\forall(i,j)\in\mathcal{N}^2}$ is jointly exchangeable.

b) The finite sequence of random vectors $\{X_i\}_{\forall i\in\mathcal{N}}$ is exchangeable.[2]

c) The equilibrium selection mechanisms adopted by the players in the network formation game is independent of the players' labels, i.e., for every permutation $\varphi$ of the labels in $\mathcal{N}$,

$$\mathbb{P}(\boldsymbol{G}\in K|\boldsymbol{X}=\boldsymbol{x},\epsilon=e)=\mathbb{P}(\boldsymbol{G}\in K^\varphi|\boldsymbol{X}=\boldsymbol{x}^\varphi,\epsilon=e^\varphi),$$

[2]Exchangeability is defined e.g., by Schervish (1995).

$\forall K \in \mathcal{K}_{\mathcal{G}}$, $\forall \boldsymbol{x} \in \mathcal{X}^n$, $\forall e \in \mathbb{R}^{n(n-1)}$ a.s., where $K^{\varphi}$ and $e^{\varphi}$ are obtained by applying $\varphi$ respectively to $K$ and $e$.

Assumption E.1 a) restricts Assumption 2 b) by imposing that the players' taste shocks are jointly exchangeable. For example, Assumption E.1 a) is satisfied if $\{\epsilon_{ij}\}_{\forall (i,j)\in \mathcal{N}^2}$ are i.i.d. Assumption 3 a) is satisfied if $\epsilon_{ij} \equiv \alpha_j + \phi_i + \xi_{ij}$ $\forall (i,j) \in \mathcal{N}^2$, with all the components being i.i.d. Assumption E.1 b) restricts Assumption 2 a) by imposing that the players' observed characteristics are exchangeable. Assumption E.1 c) restricts Assumption 2 a) by requiring that the players coordinate on the selection of an equilibrium independently of their labels. For example, Assumption E.1 c) is satisfied if the players of the *section j game* roll a die to select an outcome from the set of equilibria of the *section j game*, independently across $j$. Assumption E.1 c) is met if the players of the *section j game* select the outcome providing the highest total wealth, $\sum_{i\in\mathcal{N}\setminus\{j\}} G_{ij} \times u_{ij}(\cdot;\theta_u)$, from the equilibrium set of the *section j game*, $\forall j \in \mathcal{N}$. Assumption E.1 c) is violated if the players of the *section j game* select the outcome generating the highest payoff for player $k$ from the set of equilibria of the *section j game*, for some $j \in \mathcal{N}$.

Assumptions 2 and E.1 imply that the observed networks are jointly exchangeable. Indeed, for every permutation $\varphi$ of the labels in $\mathcal{N}$,

$$\begin{aligned}
\mathbb{P}(\boldsymbol{G} \in K, \boldsymbol{X} = \boldsymbol{x}) &= \mathbb{P}(\boldsymbol{G} \in K | \boldsymbol{X} = \boldsymbol{x})\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) \\
&= \int_{e\in\mathbb{R}^{n(n-1)}} \mathbb{P}(\boldsymbol{G} \in K | \boldsymbol{X} = \boldsymbol{x}, \epsilon = e) dF(e;\theta_\epsilon^0)\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) \\
&= \int_{e^{\varphi}\in\mathbb{R}^{n(n-1)}} \underbrace{\mathbb{P}(\boldsymbol{G} \in K^{\varphi} | \boldsymbol{X} = \boldsymbol{x}^{\varphi}, \epsilon = e^{\varphi})}_{\text{Ass. E.1 c)}} \underbrace{dF(e^{\varphi};\theta_\epsilon^0)}_{\text{Ass. E.1 a)}} \underbrace{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}^{\varphi})}_{\text{Ass. E.1 b)}} \\
&= \mathbb{P}(\boldsymbol{G} \in K^{\varphi} | \boldsymbol{X} = \boldsymbol{x}^{\varphi})\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}^{\varphi}) = \mathbb{P}(\boldsymbol{G} \in K^{\varphi}, \boldsymbol{X} = \boldsymbol{x}^{\varphi}),
\end{aligned}$$

$\forall K \in \mathcal{K}_{\mathcal{G}}$ and $\forall \boldsymbol{x} \in \mathcal{X}^n$.

Under Assumptions 2 and E.1, also the observed *section* 1, ..., *section* $n$ are jointly exchangeable. Indeed, for every permutation $\varphi$ of the labels in $\mathcal{N}$,

$$\mathbb{P}(G_{\cdot j} \in K_j, \boldsymbol{X} = \boldsymbol{x}) = \mathbb{P}(G_{\cdot\varphi(j)} \in K^{\varphi}_{\cdot\varphi(j)}, \boldsymbol{X} = \boldsymbol{x}^{\varphi}), \tag{E.1}$$

$\forall j \in \mathcal{N}$, $\forall K_j \in \mathcal{K}_{\{0,1\}^{n-1}}$ and $\forall \boldsymbol{x} \in \mathcal{X}^n$, where $K^{\varphi}_{\cdot\varphi(j)}$ is obtained by applying $\varphi$ to $K_j$.[3]

**Step 2** This step shows that, under Assumptions 1, 2, and E.1, it is sufficient to focus on the *section j game* for a $j \in \mathcal{N}$, rather than $\forall j \in \mathcal{N}$. Specifically, it is proved that, under Assumptions 1, 2, and E.1, $\Theta^{\text{o}} = \Theta^{\text{o}}_{\cdot j}$ $\forall j \in \mathcal{N}$, where

$$\begin{aligned}
\Theta^{\text{o}}_{\cdot j} \equiv \Big\{\theta \in \Theta | &\mathbb{P}(G_{\cdot j} = g_{\cdot j}, \boldsymbol{X} = \boldsymbol{x}) - \tilde{H}^l_{g_{\cdot j},\boldsymbol{x}}(\theta) \geq 0, \\
&\tilde{H}^u_{g_{\cdot j},\boldsymbol{x}}(\theta) - \mathbb{P}(G_{\cdot j} = g_{\cdot j}, \boldsymbol{X} = \boldsymbol{x}) \geq 0 \ \forall g_{\cdot j} \in \{0,1\}^{n-1}, \forall \boldsymbol{x} \in \mathcal{X}^n\Big\}.
\end{aligned} \tag{E.2}$$

*Proof.* It can be seen that if $\theta \in \Theta^{\text{o}}$, then $\theta \in \Theta^{\text{o}}_{\cdot j}$ $\forall j \in \mathcal{N}$. Hence, in what follows it is proved

[3] Notice that, under Assumption E.1 a), $\forall j \in \mathcal{N}$, the finite sequence of random variables $\{\epsilon_{ij}\}_{\forall i\in\mathcal{N}\setminus\{j\}}$ is jointly exchangeable. This is because any finite subsequence of $\{\epsilon_{ij}\}_{\forall i,j\in\mathcal{N}^2}$ is jointly exchangeable (Proposition 1.12 in Schervish, 1995)

that, under Assumptions 1, 2, and E.1, if $\theta \in \Theta^{o}_{\cdot j}$ then $\theta \in \Theta^{o}$, $\forall j \in \mathcal{N}$. This is the same as showing that if

$$\tilde{H}^{l}_{g_{\cdot j},\boldsymbol{x}}(\theta) \leq \mathbb{P}(G_{\cdot j} = g_{\cdot j}, \boldsymbol{X} = \boldsymbol{x}) \leq \tilde{H}^{u}_{g_{\cdot j},\boldsymbol{x}}(\theta) \ \forall g_{\cdot j} \in \{0,1\}^{n-1}, \forall \boldsymbol{x} \in \mathcal{X}^{n}, \tag{E.3}$$

then

$$\tilde{H}^{l}_{g_{\cdot h},\boldsymbol{x}}(\theta) \leq \mathbb{P}(G_{\cdot h} = g_{\cdot h}, \boldsymbol{X} = \boldsymbol{x}) \leq \tilde{H}^{u}_{g_{\cdot h},\boldsymbol{x}}(\theta) \ \forall g_{\cdot h} \in \{0,1\}^{n-1}, \forall h \in \mathcal{N} \setminus \{j\}, \forall \boldsymbol{x} \in \mathcal{X}^{n}, \tag{E.4}$$

$\forall \theta \in \Theta$ and $\forall j \in \mathcal{N}$.

First, under Assumptions 1, 2, and E.1, it holds that

$$\mathbb{P}(\mathcal{S}_{j,\theta_u}(\boldsymbol{X}, \epsilon_{\cdot j}) \cap K_j \neq \emptyset, \boldsymbol{X} = \boldsymbol{x}; \theta) = \mathbb{P}(\mathcal{S}_{\varphi(j),\theta_u}(\boldsymbol{X}, \epsilon_{\cdot \varphi(j)}) \cap K^{\varphi}_{\cdot \varphi(j)} \neq \emptyset, \boldsymbol{X} = \boldsymbol{x}^{\varphi}; \theta), \tag{E.5}$$

when $K_j \equiv \{g_{\cdot j}\}$ and $K_j \equiv \{0,1\}^{n-1} \setminus \{g_{\cdot j}\}$, $\forall \theta \in \Theta$, $\forall j \in \mathcal{N}$, $\forall g_{\cdot j} \in \{0,1\}^{n-1}$, $\forall \boldsymbol{x} \in \mathcal{X}^{n}$, and for every permutation $\varphi$ of the labels in $\mathcal{N}$ such that $\varphi(j) \neq j$. Indeed,

$$\begin{aligned}
&\mathbb{P}(\mathcal{S}_{j,\theta_u}(\boldsymbol{X}, \epsilon_{\cdot j}) \cap K_j \neq \emptyset, \boldsymbol{X} = \boldsymbol{x}; \theta) = \mathbb{P}(\mathcal{S}_{j,\theta_u}(\boldsymbol{X}, \epsilon_{\cdot j}) \cap K_j \neq \emptyset | \boldsymbol{X} = \boldsymbol{x}; \theta)\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) \\
&= \underbrace{\mathbb{P}(\mathcal{S}_{j,\theta_u}(\boldsymbol{x}, \epsilon_{\cdot j}) \cap K_j \neq \emptyset; \theta)}_{\text{Ass. 2 b)}} \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) \\
&= \underbrace{\mathbb{P}(\mathcal{S}_{\varphi(j),\theta_u}(\boldsymbol{x}^{\varphi}, \epsilon_{\cdot \varphi(j)}) \cap K^{\varphi}_{\cdot \varphi(j)} \neq \emptyset; \theta)}_{\text{Equilibrium sets independent of players' labels, Ass. E.1 a)}} \underbrace{\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}^{\varphi})}_{\text{Ass. E.1 b)}} \\
&= \underbrace{\mathbb{P}(\mathcal{S}_{\varphi(j),\theta_u}(\boldsymbol{X}, \epsilon_{\cdot \varphi(j)}) \cap K^{\varphi}_{\cdot \varphi(j)} \neq \emptyset | \boldsymbol{X} = \boldsymbol{x}^{\varphi}; \theta)}_{\text{Ass. 2 b)}} \mathbb{P}(\boldsymbol{X} = \boldsymbol{x}^{\varphi}) \\
&= \mathbb{P}(\mathcal{S}_{\varphi(j),\theta_u}(\boldsymbol{X}, \epsilon_{\cdot \varphi(j)}) \cap K^{\varphi}_{\cdot \varphi(j)} \neq \emptyset, \boldsymbol{X} = \boldsymbol{x}^{\varphi}; \theta).
\end{aligned}$$

Therefore, combining (E.1) with (E.5) $\forall \theta \in \Theta$, $\forall j \in \mathcal{N}$, $\forall K_j \in \mathcal{K}_{\{0,1\}^{n-1}}$, and for every permutation of labels $\varphi$ such that $\varphi(j) \neq j$, it holds that (E.3) is equivalent to (E.4). □

**Step 3** Under Assumptions 1, 2, and E.1, the inequalities in (E.2) indexed by the realisations of $(G_{\cdot j}, \boldsymbol{X})$ that are equivalent up to a permutation of the labels in $\mathcal{N}$ other than label $j$ are identical.

*Proof.* Under Assumptions 1, 2, and E.1, some inequalities in (E.3) are redundant. This is because, by (E.1) and (E.5), for every permutation $\varphi$ of the labels in $\mathcal{N}$ such that $\varphi(j) = j$,

$$\tilde{H}^{l}_{g_{\cdot j},\boldsymbol{x}}(\theta) \leq \mathbb{P}(G_{\cdot j} = g_{\cdot j}, \boldsymbol{X} = \boldsymbol{x}) \leq \tilde{H}^{u}_{g_{\cdot j},\boldsymbol{x}}(\theta),$$

is equivalent to

$$\tilde{H}^{l}_{g^{\varphi}_{\cdot j},\boldsymbol{x}^{\varphi}}(\theta) \leq \mathbb{P}(G_{\cdot j} = g^{\varphi}_{\cdot j}, \boldsymbol{X} = \boldsymbol{x}^{\varphi}) \leq \tilde{H}^{u}_{g^{\varphi}_{\cdot j},\boldsymbol{x}^{\varphi}}(\theta).$$

$\forall \theta \in \Theta$, $\forall g_{\cdot j} \in \{0,1\}^{n-1}$, and $\forall \boldsymbol{x} \in \mathcal{X}^{n}$. Such a result can be shown by mimicking the proof of Step 2. □

**Step 4** Delete from $\{0,1\}^{n-1} \times \mathfrak{X}^n$ the realisations of $(G_{\cdot 3}, \boldsymbol{X})$ generating redundant inequalities when applying all the permutations $\varphi$ of the labels in $\mathscr{N}$ such that $\varphi(3) = 3$, as explained in step 3. Let $\mathscr{W} \subset \{0,1\}^{n-1} \times \mathfrak{X}^n$ denote the resulting set. Section E.3 illustrates an algorithm to construct $\mathscr{W}$. It should be noticed that $\mathscr{W}$ is not uniquely defined because one is free to keep any of the realisations of $(G_{\cdot 3}, \boldsymbol{X})$ producing identical inequalities.

By steps 2 and 3, under Assumptions 1, 2, and E.1, conducting inference on $\Theta^{\mathrm{o}}$ is equivalent to conducting inference on

$$\Theta^{\mathrm{o}}_{\cdot 3} = \Big\{\theta \in \Theta | \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}) - \tilde{H}^l_{g_{\cdot 3}, \boldsymbol{x}}(\theta) \geq 0, \tilde{H}^u_{g_{\cdot 3}, \boldsymbol{x}}(\theta) - \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}) \geq 0 \; \forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathscr{W}\Big\}, \tag{E.6}$$

where the subscript $j$ is fixed to 3 without loss of generality.

Let $C_{g_{\cdot 3}, \boldsymbol{x}} \subset \{0,1\}^{n-1} \times \mathfrak{X}^n$ be the collection of the realisations of $(G_{\cdot 3}, \boldsymbol{X})$ giving rise to inequalities identical to the inequalities indexed by $(g_{\cdot 3}, \boldsymbol{x})$ when applying all the permutations $\varphi$ of the labels in $\mathscr{N}$ such that $\varphi(3) = 3$, as explained in step 3.[4] Following Sheng (2016), it is now proved that, under Assumptions 1, 2, and E.1, (E.6) can be rewritten as

$$\Theta^{\mathrm{o}}_{\cdot 3} = \Big\{\theta \in \Theta | \mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}}) - \tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta) \geq 0, \tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta) - \mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}}) \geq 0 \; \forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathscr{W}\Big\}, \tag{E.7}$$

where $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ is the probability that every equilibrium of the *section* 3 *game* falls in $C_{g_{\cdot 3}, \boldsymbol{x}}$ and $\tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ is the probability that at least one equilibrium of the *section* 3 *game* falls in $C_{g_{\cdot 3}, \boldsymbol{x}}$, given $\theta \in \Theta$.

*Proof.* By (E.1), under Assumptions 2 and E.1,

$$\mathbb{P}(G_{\cdot i} = g_{\cdot i}, \boldsymbol{X} = \boldsymbol{x}) = \mathbb{P}(G_{\cdot \varphi(i)} = g^{\varphi}_{\cdot \varphi(i)}, \boldsymbol{X} = \boldsymbol{x}^{\varphi}), \tag{E.8}$$

$\forall i \in \mathscr{N}$, for every permutation $\varphi$ of the labels in $\mathscr{N}$, and $\forall (g_{\cdot i}, \boldsymbol{x}) \in \{0,1\}^{n-1} \times \mathfrak{X}^n$.

By (E.8) applied for every permutation $\varphi$ of the labels in $\mathscr{N}$ such that $\varphi(3) = 3$,

$$\mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}}) = |C_{g_{\cdot 3}, \boldsymbol{x}}| \times \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}). \tag{E.9}$$

Similarly, $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ and $\tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ can be shown being equivalent to $|C_{g_{\cdot 3}, \boldsymbol{x}}| \times \tilde{H}^l_{g_{\cdot 3}, \boldsymbol{x}}$ and $|C_{g_{\cdot 3}, \boldsymbol{x}}| \times \tilde{H}^u_{g_{\cdot 3}, \boldsymbol{x}}$, respectively.

Hence,

$$\begin{aligned}
&\Big\{\theta \in \Theta | \tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta) \leq \mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}}) \leq \tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta) \; \forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathscr{W}\Big\} \\
&\underbrace{=}_{\text{(E.9)}} \Big\{\theta \in \Theta || C_{g_{\cdot 3}, \boldsymbol{x}}| \times \tilde{H}^l_{g_{\cdot 3}, \boldsymbol{x}}(\theta) \leq |C_{g_{\cdot 3}, \boldsymbol{x}}| \times \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}) \leq |C_{g_{\cdot 3}, \boldsymbol{x}}| \times \tilde{H}^u_{g_{\cdot 3}, \boldsymbol{x}}(\theta) \\
&\forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathscr{W}\Big\} = \Theta^{\mathrm{o}}_{\cdot 3}.
\end{aligned}$$

□

[4] In network theory, all the realisations of $(G_{\cdot 3}, \boldsymbol{X})$ in $C_{g_{\cdot 3}, \boldsymbol{x}}$ are called isomorphic and $C_{g_{\cdot 3}, \boldsymbol{x}}$ is an equivalence class for $(G_{\cdot 3}, \boldsymbol{X})$.

**Step 5** This step provides an estimator of $\mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}})$ that does not depend on how the players are labelled by the researcher, $\forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathscr{W}$.

Consider any $i \in \mathcal{N}$ and $(\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}) \in \{0,1\}^{n-1} \times \mathcal{X}^n$ such that there exists a permutation $\varphi$ of the labels in $\mathcal{N}$ featuring $\varphi(i) = 3$ and $(\tilde{g}^{\varphi}_{\cdot \varphi(i)}, \tilde{\boldsymbol{x}}^{\varphi}) = (g_{\cdot 3}, \boldsymbol{x})$. By (E.8),

$$\mathbb{P}(G_{\cdot i} = \tilde{g}_{\cdot i}, \boldsymbol{X} = \tilde{\boldsymbol{x}}) = \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}). \tag{E.10}$$

Consider $C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}} \subset \{0,1\}^{n-1} \times \mathcal{X}^n$. By (E.8) applied for every permutation $\varphi$ of the labels in $\mathcal{N}$ with $\varphi(i) = i$,

$$\mathbb{P}((G_{\cdot i}, \boldsymbol{X}) \in C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}) = |C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}| \times \mathbb{P}(G_{\cdot i} = \tilde{g}_{\cdot i}, \boldsymbol{X} = \tilde{\boldsymbol{x}}). \tag{E.11}$$

Hence,

$$\begin{aligned}\mathbb{P}((G_{\cdot i}, \boldsymbol{X}) \in C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}) &\underbrace{=}_{\text{(E.11)}} |C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}| \times \mathbb{P}(G_{\cdot i} = \tilde{g}_{\cdot i}, \boldsymbol{X} = \tilde{\boldsymbol{x}}) \underbrace{=}_{\text{(E.10)}} |C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}| \times \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}) \\ &\underbrace{=}_{|C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}| = |C_{g_{\cdot 3}, \boldsymbol{x}}|} |C_{g_{\cdot 3}, \boldsymbol{x}}| \times \mathbb{P}(G_{\cdot 3} = g_{\cdot 3}, \boldsymbol{X} = \boldsymbol{x}) \underbrace{=}_{\text{(E.9)}} \mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}}).\end{aligned} \tag{E.12}$$

Let

$$\hat{\mathbb{P}}_{C_{g_{\cdot 3}, \boldsymbol{x}}, M} \equiv \frac{1}{M} \sum_{m=1}^{M} \Big[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{(G_{\cdot i, m}, \boldsymbol{X}_m) \in C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}\} \Big]. \tag{E.13}$$

From (E.12) and assuming that the observed networks are i.i.d., $\hat{\mathbb{P}}_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}$ is a consistent estimator for $\mathbb{P}((G_{\cdot 3}, \boldsymbol{X}) \in C_{g_{\cdot 3}, \boldsymbol{x}})$ and does not depend on how the labels are assigned.

We suggest the following algorithm to compute $\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{(g_{\cdot i}, \boldsymbol{x}) \in C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}\}$.

1. Rewrite $(g_{\cdot 3}, \boldsymbol{x})$ by listing:

   i) $x_3$.

   ii) $g_{h3}$ $\forall h \in \mathcal{N} \setminus \{3\}$ such that $g_{h3} = 1$, disposing them with respect to $x_h$ in descending order; if there are $(h,k) \in \mathcal{N}^2$ with $h \neq k \neq 3$ such that $g_{h3} = g_{k3} = 1$ and $x_h = x_k$, then any order is allowed.

   iii) $g_{h3}$ $\forall h \in \mathcal{N} \setminus \{3\}$ such that $g_{h3} = 0$, disposing them with respect to $x_h$ in descending order; if there are $(h,k) \in \mathcal{N}^2$ with $h \neq k \neq 3$ such that $g_{h3} = g_{k3} = 0$ and $x_h = x_k$, then any order is allowed.

   iv) $x_h$ $\forall h \in \mathcal{N} \setminus \{3\}$ according to the disposition of the players adopted in the previous steps.

2. Call $A_3$ the obtained row of values.

3. $\forall i \in \mathcal{N}$ in the dataset, list:

   i) $x_i$.

   ii) $g_{hi}$ $\forall h \in \mathcal{N} \setminus \{i\}$ such that $g_{hi} = 1$, disposing them with respect to $x_h$ in descending order; if there are $(h,k) \in \mathcal{N}^2$ with $h \neq k \neq i$ such that $g_{hi} = g_{ki} = 1$ and $x_h = x_k$, then any order is allowed.

iii) $g_{hi}$ $\forall h \in \mathcal{N} \setminus \{i\}$ such that $g_{hi} = 0$, disposing them with respect to $x_h$ in descending order; if there are $(h,k) \in \mathcal{N}^2$ with $h \neq k \neq i$ such that $g_{hi} = g_{ki} = 0$ and $x_h = x_k$, then any order is allowed.

iv) $x_h$ $\forall h \in \mathcal{N} \setminus \{i\}$ according to the disposition of the players adopted in the previous steps.

4. Call $A_i$ the obtained row of values, $\forall i \in \mathcal{N}$.

Hence,

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{(g_{\cdot i}, \boldsymbol{x}) \in C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}\} \equiv \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{A_i = A_3\}.$$

**Step 6** This step provides an estimator of $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ and $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ that does not depend on how the players are labelled by the researcher, $\forall (g_{\cdot 3}, \boldsymbol{x}) \in \mathcal{W}$ and $\forall \theta \in \Theta$.

The estimation of $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ and $\tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ can be done via the simple frequency simulator proposed by McFadden (1989) and Pakes and Pollard (1989). Specifically, $\forall m \in \{1, ..., M\}$ and $\forall i \in \mathcal{N}$, $R_M$[5] realisations of the vector of preference shocks $\epsilon_{\cdot i,m}$ are drawn at random from $F_i(\cdot; \theta_\epsilon)$ and the set of equilibria of the *section i game* is found for each drawn realisation of $\epsilon_{\cdot i,m}$.[6] Then,

$$\hat{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}(\theta) \equiv \frac{1}{M}\sum_{m=1}^{M}\Big[\frac{1}{R_M \times n}\sum_{r=1}^{R_M}\sum_{i=1}^{n} \mathbb{1}\{^{\text{all equilibria of the } \textit{section i game}}_{\text{in network } m \text{ fall in } C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}}\}\Big], \tag{E.14}$$

and

$$\hat{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}(\theta) \equiv \frac{1}{M}\sum_{m=1}^{M}\Big[\frac{1}{R_M \times n}\sum_{r=1}^{R_T}\sum_{i=1}^{n} \mathbb{1}\{^{\text{at least one equilibrium of the } \textit{section i game}}_{\text{in network } m \text{ falls in } C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}}\}\Big], \tag{E.15}$$

where

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{^{\text{all equilibria of the } \textit{section i game}}_{\text{in network } m \text{ fall in } C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}}\},$$

and

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{^{\text{at least one equilibrium of the } \textit{section i game}}_{\text{in network } m \text{ falls in } C_{\tilde{g}_{\cdot i}, \tilde{\boldsymbol{x}}}}\},$$

are computed mimicking the algorithm in step 5.

As done for $\hat{\mathbb{P}}_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}$ and assuming that the observed networks are i.i.d., it can be shown that $\hat{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}(\theta)$ and $\hat{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}, M}(\theta)$ are consistent estimators for $\tilde{H}^l_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ and $\tilde{H}^u_{C_{g_{\cdot 3}, \boldsymbol{x}}}(\theta)$ and do not depend on how the labels are assigned.

[5]The subscript $M$ reminds that the number of draws of the preference shocks should increase to infinity with the sample size to avoid not vanishing simulations errors (Ciliberto and Tamer, 2009).

[6]Section 3.5 of the paper explains how, by Corollary 1, the researcher can avoid checking the equilibrium conditions for each of the $2^{n-1}$ possible realisations of $G_{\cdot i,m}$ for every drawn realisation of $\epsilon_{\cdot i,m}$.

## E.2 Construction of the confidence region

All the inequalities in (E.7) are stacked in a vector. The sample analogue of their left hand side - computed using the procedures in steps 5,6 - is denoted from now on by $\bar{b}_M(\theta)$, with generic $k$th element $\bar{b}_{k,M}(\theta)$. Let

$$S_M(\theta) \equiv \sum_k \left( \min \left\{ \frac{\sqrt{M}\bar{b}_{k,M}(\theta)}{\hat{\sigma}_{k,M}(\theta)}, 0 \right\} \right)^2,$$

where $\hat{\sigma}_{k,M}(\theta)$ is a consistent estimator of the asymptotic standard deviation of $\sqrt{M}\bar{b}_{k,M}(\theta)$. Assumptions 1, 2, and E.1, combined with an i.i.d. sampling scheme and other regularity conditions discussed by AS, imply that a valid $(1-\alpha)$ confidence region for each $\theta \in \Theta^o$ is

$$CS_{M,1-\alpha} \equiv \left\{ \theta \in \Theta \mid S_M(\theta) \leq \hat{c}_{M,1-\alpha}(\theta) \right\}, \tag{E.16}$$

where $\hat{c}_{M,1-\alpha}(\theta)$ is an estimate of the $1-\alpha$ quantile of the asymptotic probability distribution of $S_M(\theta)$, obtainable following the bootstrap procedure with hard threshold in AS. More details on the computation $\hat{c}_{M,1-\alpha}(\theta)$ are in Section E.4.

## E.3 Construction of $\mathcal{W}$

This section illustrates a way to construct the set $\mathcal{W} \subset \{0,1\}^{n-1} \times \mathcal{X}^n$ introduced earlier.

1. Rewrite each realisation $(g_{\cdot 3}, \boldsymbol{x})$ of $(G_{\cdot 3}, \boldsymbol{X}) \in \{0,1\}^{n-1} \times \mathcal{X}^n$ by listing in a row vector

   i) $x_3$.

   ii) $g_{i3}$ $\forall i \in \mathcal{N} \setminus \{3\}$ such that $g_{i3} = 1$, disposing them with respect to $x_i$ in descending order; if there are $(i,k) \in \mathcal{N}^2$ with $i \neq k \neq 3$ such that $g_{i3} = g_{k3} = 1$ and $x_i = x_k$, then any order is allowed.

   iii) $g_{i3}$ $\forall i \in \mathcal{N} \setminus \{3\}$ such that $g_{i3} = 0$, disposing them with respect to $x_i$ in descending order; if there are $(i,k) \in \mathcal{N}^2$ with $i \neq k \neq 3$ such that $g_{i3} = g_{k3} = 0$ and $x_i = x_k$, then any order is allowed.

   iv) $x_i$ $\forall i \in \mathcal{N} \setminus \{3\}$ according to the disposition of the players adopted in the previous steps.

2. For each row that is repeated once or more, delete all the duplications from the second occurrence.

3. Collect the saved rows and rearrange each of them in its original order. The resulting set is $\mathcal{W}$.

As an example, assume $n = 3$ and $\mathcal{X} \equiv \{0,1\}$. The set $\{0,1\}^2 \times \mathcal{X}^3$ is represented in Table E.1. The realisations of $(G_{\cdot 3}, \boldsymbol{X})$ giving rise to equivalent inequalities have a symbol of the same colour. Table E.2 reports in blue the rows of Table E.1 reordered according to step 1 above. It can be noticed that the algorithm described in step 1 detects all the realisations of $(G_{\cdot 3}, \boldsymbol{X})$ associated to the same colour in Table E.1. Lastly, Table E.3 lists the elements of the set $\mathcal{W}$.

| $G_{13}$ | $G_{23}$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 1 | 0 | 1 | blue |
| 1 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 1 | 0 | 0 | |
| 1 | 1 | 0 | 1 | 1 | blue |
| 1 | 1 | 0 | 0 | 1 | |
| 1 | 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 0 | 0 | |
| 1 | 0 | 1 | 1 | 1 | light green |
| 1 | 0 | 1 | 0 | 1 | green |
| 1 | 0 | 1 | 1 | 0 | magenta |
| 0 | 0 | 1 | 0 | 0 | yellow |
| 1 | 0 | 0 | 1 | 1 | orange |
| 1 | 0 | 0 | 0 | 1 | brown |
| 1 | 0 | 0 | 1 | 0 | purple |
| 1 | 0 | 0 | 0 | 0 | pink |
| 0 | 1 | 1 | 1 | 1 | light green |
| 0 | 1 | 1 | 0 | 1 | orange |
| 0 | 1 | 1 | 1 | 0 | magenta |
| 0 | 1 | 1 | 0 | 0 | purple |
| 0 | 1 | 0 | 1 | 1 | green |
| 0 | 1 | 0 | 0 | 1 | brown |
| 1 | 0 | 0 | 0 | 1 | yellow |
| 0 | 1 | 0 | 0 | 0 | pink |
| 0 | 0 | 1 | 1 | 1 | |
| 0 | 0 | 1 | 0 | 1 | red |
| 0 | 0 | 1 | 1 | 0 | |
| 0 | 0 | 1 | 0 | 0 | |
| 0 | 0 | 0 | 1 | 1 | red |
| 0 | 0 | 0 | 0 | 1 | |
| 0 | 0 | 0 | 1 | 0 | |
| 0 | 0 | 0 | 0 | 0 | |

Table E.1: This table represents the set $\{0,1\}^{n-1} \times \mathfrak{X}^n$ when $n = 3$ and $\mathfrak{X} \equiv \{0,1\}$. The realisations of $(G_{\cdot 3}, \boldsymbol{X})$ giving rise to equivalent inequalities have a symbol of the same colour in the last column.

| $G_{13}$ | $G_{23}$ | $X_1$ | $X_2$ | $X_3$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |

Table E.2: This table reports the rows of Table E.1 in their original (black) and new (blue) order according to step 1 of section E.3.

| $G_{13}$ | $G_{23}$ | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

Table E.3: This table represents the set $\mathscr{W}$ when $n = 3$ and $\mathfrak{X} \equiv \{0, 1\}$.

## E.4 Computation of the critical value

To compute the critical value, $\hat{c}_{M,1-\alpha}(\theta)$, proceed as follows.

i) For each $k$, compute $\xi_{k,M}(\theta) \equiv \frac{1}{\sqrt{\log(M)}}\sqrt{M}\frac{\bar{b}_{k,M}(\theta)}{\hat{\sigma}_{k,M}(\theta)}$.

ii) For each $k$, compute $\zeta_{k,M}(\theta) \equiv \begin{cases} 0 & \text{if } \xi_{k,M}(\theta) \leq 1 \\ \infty & \text{otherwise} \end{cases}$.

iii) Draw $B_M$ bootstrap samples.

iv) For $b = 1, ..., B_M$

(a) Compute the sample analogues of the moment inequalities defining $\Theta^o$ together with a consistent estimator of their asymptotic standard deviations, as done for the original sample. Denote the results by $\bar{b}^{\star}_{k,M,b}(\theta)$ and $\hat{\sigma}^{\star}_{k,M,b}(\theta)$ for each $k$.

(b) Compute $L_{M,b}(\theta) \equiv \sum_k \left( \min\left\{ \frac{\sqrt{M}(\bar{b}^{\star}_{k,M,b}(\theta)-\bar{b}_{k,M}(\theta))}{\hat{\sigma}^{\star}_{k,M,b}(\theta)} + \zeta_{k,M}(\theta), 0 \right\}\right)^2$.

v) Define $\hat{c}_{M,1-\alpha}(\theta)$ as the $(1-\alpha)$ sample quantile of $\{L_{M,b}(\theta)\}_{b=1}^{B}$.

## E.5 Construction of the initial grid of parameter values

One difficulty with conducting inference on sets is scanning over a multi-dimensional parameter space. In practice, what the researcher can do is exploring the parameter space around the global infimum of $S_M(\theta)$ in some rational way. For the empirical illustration in this paper, we follow the simulated annealing method proposed by Ciliberto and Tamer (2009):

i) List many starting values for $\theta$, one of which has all entries equal to zero, others are constructed using the results of simple probits.

ii) From every starting value for $\theta$, minimise $S_M(\theta)$ by running the simulated annealing algorithm in Matlab and save each parameter value encountered in the course of the algorithm.

iii) Look at the saved parameter values. This collection is the initial grid of parameter values.

## E.6 Monte Carlo exercises

This section reports the results of some Monte Carlo experiments. We consider two payoff specifications:

$$u_i(\boldsymbol{G}, \boldsymbol{X}, \epsilon; \theta_u) \equiv \sum_{j=1}^{N} G_{ij} \times \left[\beta \times |X_i - X_j| + \delta \times \sum_{k\neq i}^{N} G_{kj} + \epsilon_{ij}\right], \tag{E.17}$$

with true parameter vector $\theta_0 \equiv (\beta_0, \delta_0) = (0.8, -0.9)$ (substitution effects), and

$$u_i(\boldsymbol{G}, \boldsymbol{X}, \epsilon; \theta_u) \equiv \sum_{j=1}^{N} G_{ij} \times \left[\beta \times |X_i - X_j| + \frac{\delta}{N-2} \times \sum_{k\neq i}^{N} G_{kj} + \epsilon_{ij}\right], \tag{E.18}$$

with true parameter vector $\theta_0 \equiv (\beta_0, \delta_0) = (-0.5, 0.4)$ (complementary effects). To generate the data, we impose $\{X_i\}_{\forall i \in \mathcal{N}}$ i.i.d., $X_i \sim \text{Unif}(\{0,1\})$, $\epsilon_{ij} \equiv u_{ij} + u_i + u_j \ \forall (i,j) \in \mathcal{N}^2$, all the taste shocks i.i.d. standard normal independent of $\boldsymbol{X}$. Lastly, in case of multiple equilibria, we let players select one outcome uniformly at random from the equilibrium set of each local game, independently across local games.

First, we plot the empirical probability distribution of $\frac{1}{M}S_M(\theta_0)$ for different values of $N, M$. In Figure E.1, the three panels are obtained from specification (E.17) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$. In Figure E.2, the three panels are obtained from specification (E.18) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$. In all the panels the number of draws of taste shocks to compute the integrals entering the moment inequalities is set equal to $\frac{M}{2}$. As expected, the empirical probability distribution of $\frac{1}{M}S_M(\theta_0)$ shrinks around zero as $M$ increases for each value of $N$.

Second, we calculate the coverage frequency of $\theta_0$ in the 95% confidence region, $CS_{M,0.95}$. Specifically, we count the fraction of Monte Carlo experiments such that $\theta_0$ belongs to $CS_{M,0.95}$ over 500 replications, for various values of $N$ and $M$. Table E.4 is obtained from specification (E.17) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$. Table E.5 is obtained from specification (E.18) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$. The number of draws of taste shocks to compute the integrals entering the moment inequalities and the number of bootstrap samples to obtain the critical values are set both equal to $\frac{M}{2}$. As expected, the coverage frequency adjusts to a value greater or equal than 0.95 as $M$ increases for each value of $N$.

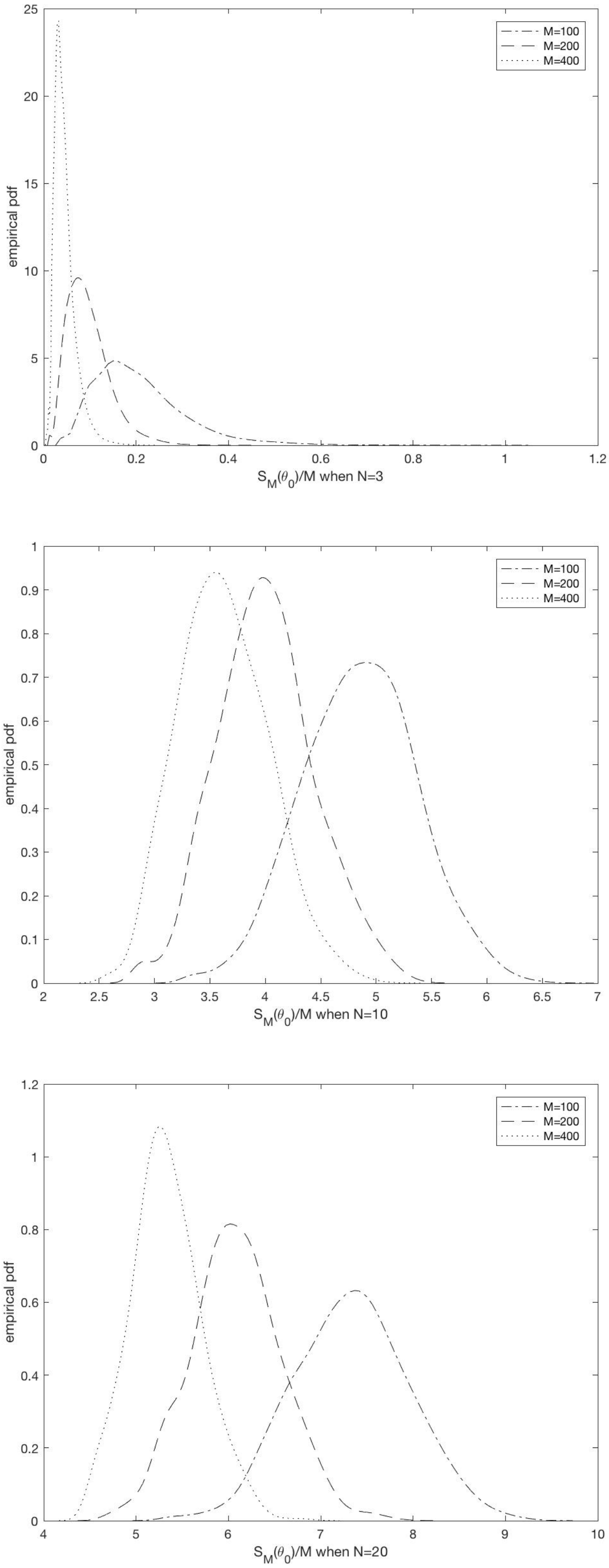


Figure E.1: The empirical probability distribution of $\frac{1}{M} S_M(\theta_0)$ is plotted for various values of $N$ and $M$. The three panels are obtained from specification (E.17) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$.

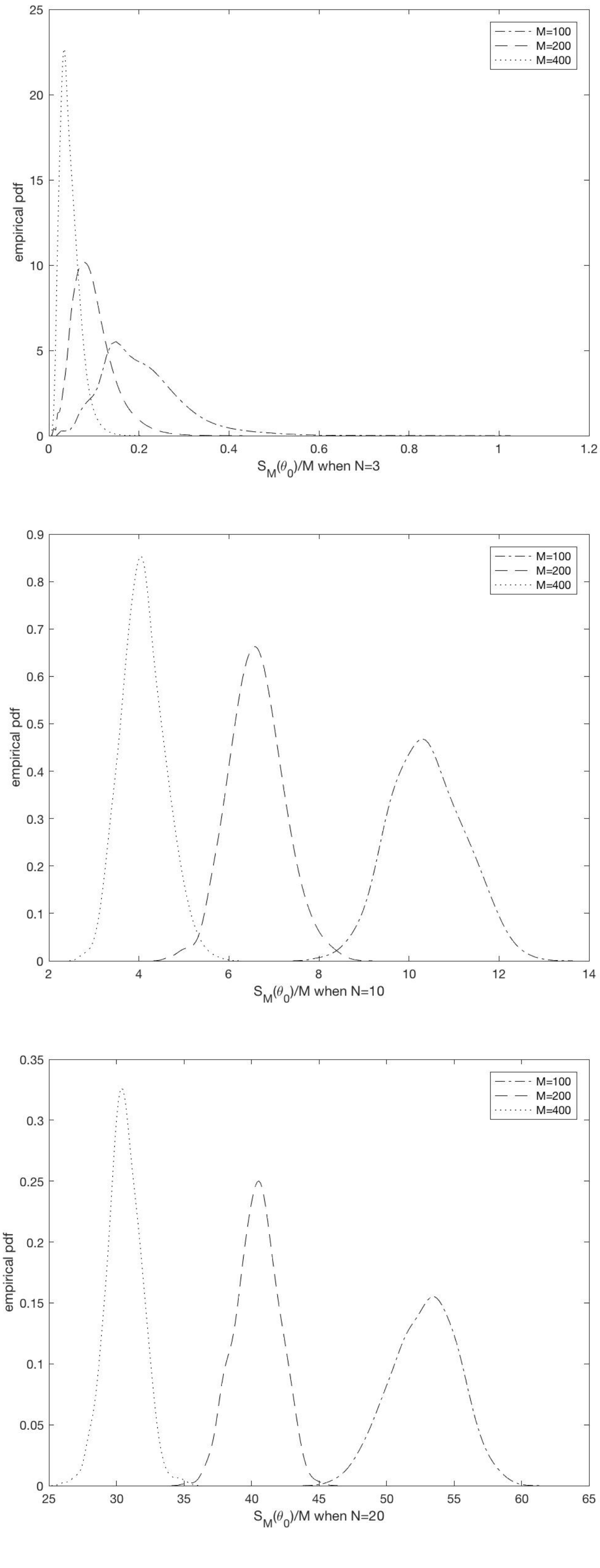


Figure E.2: The empirical probability distribution of $\frac{1}{M}S_M(\theta_0)$ is plotted for different values of $N, M$. The three panels are obtained from specification (E.18) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$.

| | $M = 100$ | $M = 200$ | $M = 400$ |
|---|---|---|---|
| $N = 3$ | 0.948 | 0.970 | 0.974 |
| $N = 10$ | 1 | 1 | 1 |
| $N = 20$ | 1 | 1 | 1 |

Table E.4: This table reports the fraction of Monte Carlo experiments such that $\theta_0$ belongs to the 95% confidence region, $CS_{M,95}$, over 500 replications. The table is obtained from specification (E.17) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$.

| | $M = 100$ | $M = 200$ | $M = 400$ |
|---|---|---|---|
| $N = 3$ | 0.942 | 0.982 | 0.966 |
| $N = 10$ | 1 | 1 | 0.994 |
| $N = 20$ | 1 | 1 | 1 |

Table E.5: This table reports the fraction of Monte Carlo experiments such that $\theta_0$ belongs to the 95% confidence region, $CS_{M,95}$, over 500 replications. The table is obtained from specification (E.18) by setting $N = 3, 10, 20$ and $M = 100, 200, 400$.

# F Empirical application

## F.1 Data construction and cleaning

In order to extract and merge the information from the Registro Imprese database, each firm is uniquely identified by combining its Chamber of Commerce's territorial province, R.E.A code, and tax code. The R.E.A. code is a number assigned to each company when enrolling at the Registro Imprese database and stands for *Repertorio Economico Amministrativo*. The tax code is a numeric code of 16 digits.

Each board member is uniquely identified by an individual code, which is an alphanumeric code of 16 characters, similar to the Social Security Number in the US or the National Insurance Number in the UK.

In order to merge the information from the Registro Imprese database with that from the Cerved database, we use the firms' tax codes.

Industries are constructed by considering the firms' principal lines of activity provided by the 5 digit-ATECO 2002 code, which is obtained from the Registro Imprese database. The ATECO 2002 code is an alpha-numeric code with varying degrees of detail. It is developed in five levels: sections (letter), subsections (two letters, optional), divisions (2 digits), groups (3 digits), classes (4 digits), and categories (5 digits). For example,

- A: Agriculture, hunting and fishing
- 01: Agriculture, hunting and related service activities
- 01.1: Crops
- 01.11: Growing of cereals and other arable crops
- 01.11.1: Growing of cereals (rice included)
- 01.11.2: Growing of oil seeds
- ...

In 2008 the ATECO 2002 code was replaced by the ATECO 2007 code, whose structure preserves the same general characteristics of its predecessor. However, we use the ATECO 2002 code because its data quality is remarkably higher for the year 2010.

Industries composed of 1 or 2 firms are dropped because the model requires $N \geq 3$.

## F.2 Some descriptive statistics

Some descriptive statistics for industry size, total assets, and return on equity are in Table F.1. Some network summary statistics are in Table F.2.

| | Mean | Standard deviation | Min | Max | $[0.25; 0.50; 0.75]$ quantiles | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| $N$ | 6.733 | 3.483 | 3 | 15 | $[4; 6; 9]$ | 0.812 | 2.645 |
| TA ($\times 10^6$ €) | 117.281 | $1,568.453$ | 0.067 | $73,916.239$ | $[6.998; 15.653; 39.984]$ | 41.471 | $1,903.552$ |
| ROE (%) | 1.267 | 24.589 | $-128.410$ | 69.820 | $[-2.382; 2.360; 11.402]$ | $-1.600$ | 9.071 |

Table F.1: This table reports some descriptive statistics for industry size ($N$), total assets (TA), and return on equity (ROE). All the numbers are obtained by computing the mean, standard deviation, minimum, maximum, $0.25, 0.50, 0.75$ quantiles, skewness, and kurtosis of $N$, TA, and ROE within each industry and, then, averaging across industries.

| | Mean | Standard deviation | Min | Max | $[0.25; 0.50; 0.75]$ quantiles | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| Density | 0.005 | 0.026 | 0 | 0.333 | $[0; 0; 0]$ | 8.462 | 88.322 |
| Average degree | 0.023 | 0.096 | 0 | 1 | $[0; 0; 0]$ | 5.905 | 45.181 |
| % Isolated nodes | 97.666 | 8.758 | 33.333 | 100 | $[100; 100; 100]$ | $-4.299$ | 22.587 |
| Number of links | 0.163 | 0.617 | 0 | 6 | $[0; 0; 0]$ | 4.859 | 32.750 |

Table F.2: This table reports some descriptive statistics for the density, average degree, percentage of isolated nodes, and number of links of the networks observed in the data. All the numbers are obtained by computing the mean, standard deviation, minimum, maximum, $0.25, 0.50, 0.75$ quantiles, skewness, and kurtosis of the network statistics within each industry and, then, averaging across industries.

# References


Andrews, D.W.K., and X. Shi (2013): "Inference Based on Conditional Moment Inequalities", *Econometrica*, 81(2), 609–666.

Andrews, D.W.K., and X. Shi (2017): "Inference Based on Many Conditional Moment Inequalities", *Journal of Econometrics*, 196(2), 275–287.

Andrews, D.W.K., and G. Soares (2010): "Inference for Parameters Defined by Moment Inequalities Using Generalized Moment Selection", *Econometrica*, 78(1), 119–157.

Bugni, F., I. Canay, and X. Shi (2017): "Inference for Subvectors and Other Functions of Partially Identified Parameters in Moment Inequality Models", *Quantitative Economics*, 8(1), 1–38.

Bugni, F., M. Caner, A.B. Kock, and S. Lahiri (2016): "Inference in Partially Identified Models with Many Moment Inequalities Using Lasso", Working Paper arXiv:1604.02309.

Chernozhukov, V., H. Hong, and E. Tamer (2007): "Estimation and Confidence Regions for Parameter Sets in Econometric Models", *Econometrica*, 75(5), 1243–1284.

Chernozhukov, V., S. Lee, and A.M. Rosen (2013): "Intersection Bounds: Estimation and Inference", *Econometrica*, 81(2), 667–737.

Ciliberto, F., and E. Tamer (2009): "Market Structure and Multiple Equilibria in Airline Markets", *Econometrica*, 77(6), 1791–1828.

Kaido, H., F. Molinari, and J. Stoye (2019): "Confidence Intervals for Projections of Partially Identified Parameters", *Econometrica*, 87(4),1397–1432.

Kallenberg, O. (2005): *Probabilistic Symmetries and Invariance Principles*, Springer.

Pakes, A., J. Porter, K. Ho, and J. Ishii (2015): "Moment Inequalities and Their Application", *Econometrica*, 83(1), 315–334.

Sheng, S. (2016): "A Structural Econometric Analysis of Network Formation Games", Working Paper.

Schervish, O. (1995): *Theory of Statistics*, Springer.

Romano, J.P., and A. Shaikh (2008): "Inference for Identifiable Parameters in Partially Identified Econometric Models", *Journal of Planning and Statistical Inference*, 138(9), 2786–2807.

Romano, J.P., and A. M. Shaikh (2010): "Inference for the Identified Set in Partially Identified Econometric Models", *Econometrica*, 78(1), 169–211.

Romano, J., A. Shaikh, and M. Wolf (2014): "A Practical Two-Step Method for Testing Moment Inequalities with an Application to Inference in Partially Identified Models", *Econometrica*, 82(5), 1979–2002.

Rosen, A.M. (2008): “Confidence Sets for Partially Identified Parameters that Satisfy a Finite Number of Moment Inequalities”, *Journal of Econometrics*, 146(1), 107–117.