

IDENTIFICATION IN DISCRETE CHOICE MODELS WITH IMPERFECT INFORMATION*

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Abstract

We study identification of preferences in static single-agent discrete choice models where decision makers may be imperfectly informed about the state of the world. We leverage the notion of one-player Bayes Correlated Equilibrium by Bergemann and Morris (2016) to provide a tractable characterization of the sharp identified set. We develop a procedure to practically construct the sharp identified set when the state of the world is continuous following a sieve approach, and provide sharp bounds on counterfactual outcomes of interest. We use our methodology and data on the 2017 UK general election to estimate a spatial voting model under weak assumptions on agents' information about the returns to voting. Counterfactual exercises quantify the consequences of imperfect information on the well-being of voters and parties.

KEYWORDS: Discrete choice model, Bayesian Persuasion, Bayes Correlated Equilibrium, Incomplete Information, Partial Identification, Moment Inequalities, Spatial Model of Voting.

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1 Introduction

A fundamental issue in the empirical analysis of decision problems is the presence of frictions that prevent agents from learning the payoffs associated with the available alternatives. When estimating parameters and making counterfactual predictions, it is common to make strong assumptions about agents' beliefs, which can weaken the credibility of the results. In particular, most of the applied literature either imposes perfect information, or incorporates information frictions by fully specifying agents' beliefs, as in search models (Mehta, Rajiv, and Srinivasan, 2003; Honka and Chintagunta, 2016; Ursu, 2018), models with rational inattention (Caplin and Dean, 2015; Matějka and McKay, 2015; Fosgerau, Melo, de Palma, and Shum, 2020; Csaba, 2018; Caplin, Dean, and Leahy, 2019; Brown and Jeon, 2020), and models with preferences for risk (for a review, see Barseghyan, Molinari, O'Donoghue, and Teitelbaum, 2018). Instead, this paper develops a methodology to identify preferences and counterfactual outcomes from cross-sectional choice data that imposes weak restrictions on the agents' beliefs and, hence, is robust to whether agents are perfectly or partially informed.

We consider a large class of static single-agent discrete choice models, where the decision maker (DM) has to choose an alternative from a finite set. The payoff generated by each alternative depends on the state of the world, which is randomly determined by nature. The DM has a prior on the state of the world. Moreover, the DM can refine their prior upon reception of a private signal representing the DM's information structure. This information structure can range from full revelation of the state of the world to no information whatsoever, depending on the latent frictions encountered by the DM in the learning process. The DM uses the acquired information structure to update their prior and obtain a posterior through the Bayes' rule. Lastly, the DM chooses an alternative maximising their expected payoff, where the expectation is computed via the posterior. Under additional assumptions on information structures, this framework accommodates additive random utility discrete choice models, discrete choice models with risk aversion, discrete choice models with rational inattention, and some discrete choice models with search. Our objective is (partially) identifying preferences and counterfactual outcomes while remaining agnostic about information structures.

The model just described is a game against nature (Milnor, 1951). That is, it is a 1-player

game in which a single self-interested player must choose a strategy. The player’s payoff depends on their own strategy and the realization of the state of the world, which is decided by a totally disinterested nature. Thus, we can use results from the theoretical literature on N -player games with $N \geq 1$ and weak assumptions on information structures to characterize the sharp identified set for the payoff parameters. In particular, we revisit our framework through the lens of 1-player Bayes Correlated Equilibrium (Kamenica and Gentzkow, 2011; Bergemann and Morris, 2013; 2016). The concept of 1-player Bayes Correlated Equilibrium is based on a theoretical setting where an omniscient mediator makes incentive-compatible recommendations to DMs as a function of the state of the world and compatible with their priors. If DMs follow such recommendations, the resulting distribution of choices is a 1-player Bayes Correlated Equilibrium. A fundamental result in the theoretical literature on robust predictions (Theorem 1, Bergemann and Morris, 2016) is that the set of optimal strategies predicted by our model under a large range of possible information structures is equivalent to the collection of model-implied choice probabilities under the notion of 1-player Bayes Correlated Equilibrium. Further, the latter collection is a convex set defined by linear equalities and inequalities. Therefore, determining whether a given parameter value belongs to the sharp identified set consists of solving a linear program, which is a well-understood and computationally tractable problem.

Observe that the collection of model-implied choice probabilities under the notion of 1-player Bayes Correlated Equilibrium can also be written as the Aumann expectation of the random set of 1-player Bayes Correlated Equilibria. Therefore, the above characterization of the sharp identified set is equivalent to the one provided by Beresteanu, Molchanov, and Molinari (2011). A distinctive feature of our framework is that this Aumann expectation is defined by linear equalities and inequalities and, therefore, can be computed by solving a linear program.

We make three main methodological contributions. On first impression, one may think of our framework as a special instance of 2-player games, where nature is the second player, thus making the analysed setting a sub-case of Syrgkanis, Tamer, and Ziani (2021) and Magnolfi and Roncoroni (2022), which study close questions for many-player games. However, there is a crucial difference between our model and 2-player games. While in the latter both players choose actions based on best-response mechanisms, in the former nature selects the state of the world

randomly without following any specific rationale. Therefore, our model is non-nested with 2-player games, which has implications for identification power. In particular, on the one hand, we show that our framework has no empirical content without variation from exogenous covariates or (non)parametric restrictions on the payoff function and prior, as is the case in several many-player games. On the other hand, we argue that, even in the presence of such additional restrictions, our framework is expected to generally provide less informative bounds on the payoff parameters than many-player games, due to the random behaviour of nature, which cannot be “influenced” by shifting covariates to extreme values, as instead is commonly done in many-player games to get tight (possibly, singleton) bounds.

Second, we develop a formal procedure to practically construct the sharp identified set for the payoff parameters when the state of the world is continuous. In such a case, a 1-player Bayes Correlated Equilibrium is an infinite-dimensional object, and, thus, the linear program to solve for each candidate parameter value is also infinite-dimensional. Previous papers simplify the analysis by assuming that the state of the world is discrete, or allowing for a continuous state of the world but, in practice, discretising its support in some arbitrary bins to operationalise the linear programming procedure. Here, instead, we propose a sieve approximation of the program along the lines of Han and Yang (2022) in their study of treatment effects. We test this procedure in simulations and implement it in the empirical application.

Third, we characterize sharp bounds on the counterfactual choice probabilities when agents receive information about the state of the world via a policy program. This is an important question in the empirical literature on single-agent decision problems across different fields and is largely absent in the literature on many-player games. To cite just a few examples, see Hastings and Tejeda-Ashton (2008) on retirement fund options in Mexico, Hastings and Weinstein (2008) and Bettinger, et al. (2012) on school choice, and Kling, et al. (2012) on Medicare Part D prescription drug plans. This question is typically answered using field experiments which can be very costly. Our characterization represents a powerful result because it allows the analyst to assess the effect of programs of information provision before conducting such interventions. We also provide sharp bounds on the maximum potential welfare cost of limited information which may keep agents from choosing their first best.

We use our methodology to study voting behaviour in the UK. We consider the spatial model of voting, which is an important framework in political economy to explain individual preferences for parties (Downs 1957; Black, 1958). This model postulates that an agent has a most preferred policy and votes for the party whose position is closest to their ideal. In empirical analysis, it is typically implemented by estimating a classical parametric discrete choice model with perfect information (Alvarez and Nagler, 1995; 1998; 2000; Alvarez, Nagler, and Bowler, 2000). However, in reality, uncertainty pervades voting (Shepsle, 1972; Weisberg and Fiorina, 1980; Enelow and Hinich, 1981; Baron, 1994; Matsusaka 1995; Carpini and Keeter, 1996; Lupia and McCubbins, 1998; Feddersen and Pesendorfer, 1999; Matějka and Tabellini, 2019). That is, voters may be aware of their own and the parties' attitudes towards some popular issues, but they might be less prepared on how they themselves and the parties stand towards more technical or less debated topics, and on the traits of the candidates other than those publicly advertised. Further, their competence on these matters is likely to be arbitrarily different depending on, for example, political sentiment, civic sense, attentional limits, media exposure, and candidates' candor.

Despite the acknowledgement of the central role played by the sophistication of voters in determining voting patterns, only a few empirical works have attempted to take it into account while estimating a spatial voting framework (Aldrich and McKelvey, 1977; Bartels, 1986; Palfrey and Poole, 1987; Franklin, 1991; Alvarez, 1998; Degan and Merlo, 2011). This has been done by exogenously and parametrically modelling how information frictions affect the perceptions of DMs about the returns to voting (for instance, via an additive, exogenous, and parametrically distributed evaluation error in the payoffs), or by parametrically specifying the probability of being informed versus uninformed when voting. Instead, our methodology permits us to incorporate voter uncertainty under weak assumptions on the latent, heterogeneous, and potentially endogenous process followed by voters to gather and evaluate information.

In particular, we focus on a setting where the state of the world consists of distances between the voters and the parties' ideological positions on a few popular policy issues, and of voter-party-specific taste variables capturing evaluations of the candidates' qualities and the parties' opinions on more complicated and less media-covered topics. We assume that each voter observes the realization of the former, but may be uncertain about the realization of the latter. We estimate

the model using data from the British Election Study, 2017: Face-to-Face Post-Election Survey (Fieldhouse, et al., 2018) on the UK general election held on 8 June 2017. We compare our findings with the results one gets under the standard assumption that all agents are perfectly informed about the returns to voting. Several conclusions on the payoff parameters achieved under the complete information assumption are not unambiguously corroborated when we remain agnostic about information structures.

We use our characterization of counterfactual bounds to robustly assess to what extent imperfect information affects the well-being of voters and parties. We imagine an omniscient mediator implementing a policy that gives voters perfect information about the state of the world. We simulate the counterfactual vote shares and study how they change compared to the factual scenario. This question has been debated at length in the literature. Political scientists have often answered it by arguing that a large population composed of possibly uninformed citizens act as if it was perfectly informed (for a review, see Bartels, 1996). Carpini and Keeter (1996), Bartels (1996), and Degan and Merlo (2011) provide quantitative evidence to disconfirm such claims; the first two by using auxiliary data on the level of information of the survey respondents as rated by the interviewers or assessed by test items, and the latter by parametrically specifying the probability that a voter is informed. We contribute to this literature by providing a way to construct counterfactual vote shares under perfect information, which neither requires the difficult task of measuring voters' knowledge level in the factual scenario, nor imposes parametric assumptions on the probability that a voter is informed. We find that voters benefit from full information as it leads to a considerable drop in the abstention rate. We also find that transparency harms the two historically dominant parties, i.e., the Conservative Party and the Labour Party, and favours the other minor parties, i.e., the Liberal Democrats and the Green Party. This suggests that some payoff-relevant information is unobserved by voters, and the biggest parties in the British political scene benefit from such uncertainty. Moreover, we quantify the maximum voters' welfare cost of limited information and find that it is comparable in magnitude to reducing the left-right ideological distance from a given party by around three points.

In addition to studying the impact of information provision, we investigate how the parties' welfare changes when they modify their ideological positions on popular policy issues. We adapt to

our setting Theorem 1 of Bergemann, Brooks, and Morris (2022), which permits us to answer such a question while holding fixed the voters’ information structures in the counterfactual scenario. In agreement with several post-election studies, we find that, by holding a strong left ideological position about tax and social care, the Labour Party gained numerous votes during the election campaign.

The remainder of the paper is organised as follows. Section 2 describes the model. Section 3 discusses identification. Section 4 presents some simulations. Section 5 illustrates the empirical application. Finally, section 6 concludes. The proofs are in Appendix B.

In what follows, we shorten Bergemann and Morris (2016) as BM16, Syrgkanis, Tamer, and Ziani (2021) as STZ21, and Magnolfi and Roncoroni (2022) as MR22. We refer to 1-player Bayes Correlated Equilibrium as 1BCE. Given a random variable Z with support \mathcal{Z} , P_Z denotes its probability mass function (if Z is discrete) or density (if Z is continuous). $\Delta(\mathcal{Z})$ denotes the set of all probability mass functions or densities with support contained in \mathcal{Z} .

2 The model

We consider a DM who faces the problem of choosing an alternative from a finite set, \mathcal{Y} . There is an unknown state of the world, V , with support \mathcal{V} , that enters directly in the DM’s utility, $u : \mathcal{Y} \times \mathcal{V} \rightarrow \mathbb{R}$. $u \in \mathcal{U}$, where \mathcal{U} is the set of all functions mapping $\mathcal{Y} \times \mathcal{V}$ to \mathbb{R} . The DM does not observe the realization of V and has a prior belief on it, P_V . Before making a choice, the DM has an opportunity to learn more about the state of the world and resolve some uncertainty about the payoffs generated by the alternatives. Formally, the DM can refine their prior on V upon reception of a private signal, T , with support \mathcal{T} and distribution $P_{T|V}(\cdot|v)$ conditional on $V = v$. We denote by $\mathcal{P}_{T|V}$ the family of the signal’s conditional distributions for each $v \in \mathcal{V}$, i.e., $\mathcal{P}_{T|V} := \{P_{T|V}(\cdot|v) : v \in \mathcal{V}\}$. The DM uses $\mathcal{P}_{T|V}$ and the received signal realization, t , to update their prior on V via Bayes’ rule and obtains a posterior, $P_{V|T}(\cdot|t)$. The DM chooses alternative $y \in \mathcal{Y}$ maximising their expected utility computed under the posterior, $\int_{\mathcal{V}} u(y, v)P_{V|T}(v|t)dv$. If there is more than one maximising alternative, the DM applies some tie-breaking rule.

The informativeness of T about V (in the Blackwell sense) is inherently related to the frictions

potentially encountered by the DM while investigating the state of the world.¹ These frictions can stem from various sources, such as attentional and cognitive limits, financial constraints, spatial and temporal boundaries, and cultural and personal biases. If the DM faces no information frictions, they may process a signal revealing the realization of V and discover the payoffs with certainty. Instead, if the DM experiences considerable information frictions, they may process a signal adding nothing to their prior on V . A signal whose informativeness is between such two extremes is plausible as well. In a typical empirical application, the information frictions the DM encounters are not observed by the researcher. Hence, we proceed without assumptions on \mathcal{T} and $\mathcal{P}_{T|V}$.

We now provide a more compact representation of our framework. Following the terminology of BM16, we define the *baseline decision problem* faced by the DM as $G := \{\mathcal{Y}, \mathcal{V}, u, P_V\}$. We also define the *information structure* processed by the DM as $S := \{\mathcal{T}, \mathcal{P}_{T|V}\}$. G represents what the DM knows before processing any signal. S consists of the additional information the DM learns about V , together with the received realization of the signal. S belongs to \mathcal{S} , which is the set of all possible information structures processed by the DM. \mathcal{S} contains the information structure giving complete information (hereafter *complete information structure*), the information structure giving no information in addition to the prior (hereafter, *null information structure*), and any information structure whose informativeness is between those two extremes. The pair $\{G, S\}$ constitutes the *augmented decision problem* faced by the DM.

We denote by Y the DM's choice. An optimal (mixed) strategy in the augmented decision problem $\{G, S\}$ is a distribution of Y conditional on T , $\mathcal{P}_{Y|T} := \{P_{Y|T}(\cdot|t) : t \in \mathcal{T}\}$, such that, for each $t \in \mathcal{T}$, the DM maximises their expected utility by choosing any alternative $y \in \mathcal{Y}$ featuring $P_{Y|T}(y|t) > 0$.²

Observe that the model just described is a *game against nature* (Milnor, 1951). That is, it is a 1-player game in which a single self-interested player must choose a strategy. Nature is indifferent among outcomes, has no payoff, and chooses V through randomisation. The player's payoff depends on their own strategy and the realization of V . At first glance, one may think of our framework as a special instance of 2-player games, where nature is the second player.

¹Blackwell (1951; 1953) provides a rank-ordering of information structures in terms of their informativeness.

²A mixed strategy arises if there are ties.

However, there is a key difference between our framework and 2-player games. While in the latter both players choose actions based on best-response mechanisms, in the former nature chooses V randomly from P_V without following any specific rationale. This makes our framework non-nested with 2-player games. In Section 3.3, we illustrate the implications of this fundamental distinction for identification.

Our framework encompasses several settings of empirical interest, primitives of which are typically estimated under strong assumptions on \mathcal{T} and $\mathcal{P}_{T|V}$. For example, it includes additive random utility discrete choice models (Logit, Nested Logit, Mixed Logit, Probit, etc.), discrete choice models with preferences for risk (Barseghyan, Molinari, O’ Donoghue, and Teitelbaum, 2013; Barseghyan, Molinari, and Teitelbaum, 2016), models with rational inattention (Caplin and Dean, 2015; Matějka and McKay, 2015; Csaba, 2018; Caplin, Dean, and Leahy, 2019; Brown and Jeon, 2020; Fosgerau, Melo, de Palma, and Shum, 2020), and some search models (Hébert and Woodford, 2018; Morris and Strack, 2019). See Appendix A for more details. Also, see Appendix D for a connection with the consideration set literature.

3 Identification

3.1 Data generating process and characterization of the identified set

We assume that the utility function, u , and prior, P_V , belong to parametric classes, $\{u(\cdot; \theta_u)\}_{\theta_u \in \Theta_U}$ and $\{P_V(\cdot; \theta_V)\}_{\theta_V \in \Theta_V}$, indexed by the finite-dimensional structural parameters $\theta_u \in \Theta_U$ and $\theta_V \in \Theta_V$, respectively. Let $\theta := (\theta_u, \theta_V) \in \Theta := \Theta_U \times \Theta_V$ denote a generic parameter vector and $\theta_0 := (\theta_{0,u}, \theta_{0,V})$ denote the true parameter vector. Let Y_1, \dots, Y_n be an i.i.d. sample of choices, where each choice is the outcome of the augmented decision problem, $\{G(\theta_0), S_1\}, \dots, \{G(\theta_0), S_n\}$, respectively.

We do not know the exact information structures, S_1, \dots, S_n , that were processed in each of these decision problems and remain agnostic about those. In particular, we allow S_i to be different from S_j for each $i \neq j$, implying that the empirical distribution of choices, \mathbb{P}_Y , is a mixture of optimal strategies over various information structures. This heterogeneity embeds the fact that different agents could encounter different information frictions and hence process more

or less informative signals. We treat the information structures processed by DMs as nuisance parameters and study the question of identifying θ_0 and counterfactuals of interest from \mathbb{P}_Y .³

All DMs are assumed to rely on a common prior, $P_V(\cdot; \theta_{0,V})$. Some heterogeneity of priors across DMs can be introduced by including discrete payoff-relevant variables, (X, ϵ) , that are observed by DMs together with the signal and correlated with V . In that case, each DM has a prior $P_{V|X,\epsilon}(\cdot; |x, e; \theta_{0,V})$ conditional on $(X, \epsilon) = (x, e)$. See Section 3.2 and Appendix C on how to add (X, ϵ) to our framework.

In certain settings, some or all the components of V are observed by the researcher. For example, in models of insurance plans, the researcher often has data on the ex-post claim experience of the agents in the sample (see Example 2 in Appendix A). In those cases, $\theta_{0,V}$ could be identified directly from such additional data. In our general discussion below, we focus on the scenario where V is unobserved to the researcher. This is the case considered in our empirical application on voting behaviour.

Intuitively, the identified set of θ_0 is the set of θ s for which the model predicts a distribution of Y that matches \mathbb{P}_Y . Let $\mathcal{R}(\theta, S)$ be the set of optimal strategies of $\{G(\theta), S\}$ and $\mathcal{R}(\theta)$ be the set of model-implied choice probabilities while remaining agnostic about information structures. That is,

$$\mathcal{R}(\theta) := \text{Conv} \left\{ P_Y \in \Delta(\mathcal{Y}) : P_Y(y) = \int_{\mathcal{T}} \int_{\mathcal{V}} P_{Y|T}(y|t) P_{T|V}(t|v) P_V(v; \theta_V) dv dt \quad \forall y \in \mathcal{Y}, \right. \\ \left. P_{Y|T} \in \mathcal{R}(\theta, S), S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S} \right\}, \quad (1)$$

where we have used the fact that Y is independent of V conditional on T . Convexification (via the convex hull operator, $\text{Conv}\{\cdot\}$) allows us to include in $\mathcal{R}(\theta)$ distributions of Y that are mixtures of optimal strategies over various information structures. This ensures the heterogeneity of information structures in the cross-section, as discussed above.⁴ The identified set of θ_0 is

³It is implicit in our discussion that we also remain agnostic about tie-breaking rules and allow them to be heterogeneous across DMs.

⁴To understand the convexification step better, note that the information structures in our framework are econometrically similar to the equilibrium selection mechanisms in incomplete many-player games (Tamer, 2003; Ciliberto and Tamer, 2009). In the former, convexification allows the information structures to differ across DMs. In the latter, convexification allows the equilibrium selection mechanisms to differ across markets. See also STZ21 and MR22 about convexification.

defined as,

$$\Theta^* := \{\theta \in \Theta : \mathbb{P}_Y \in \mathcal{R}(\theta)\}.$$

The above definition of Θ^* is not helpful in practice. This is because constructing $\mathcal{R}(\theta)$ following (1) is infeasible due to the necessity of exploring the large class \mathcal{S} , which contains infinite-dimensional objects. We overcome this issue by recalling that our decision problem is a 1-player game (game against nature), as discussed in Section 2. Hence, we can use results from the theoretical literature on N -player games with $N \geq 1$ and weak assumptions on information to give a simpler characterization of $\mathcal{R}(\theta)$ and, in turn, Θ^* .

In particular, we consider the notion of 1BCE (Kamenica and Gentzkow, 2011; Bergemann and Morris, 2013; BM16). This notion refers to a theoretical setting where an omniscient mediator makes incentive-compatible recommendations to the DM as a function of the state of the world and consistent with the DM's prior. If the DM follows such recommendations, the resulting distribution of choices is a 1BCE. A fundamental result in the theoretical literature on robust predictions (Theorem 1, BM16) is that the set of optimal strategies that arise from adding an arbitrary information structure to the baseline decision problem $G(\theta)$ is equivalent to the set of 1BCE of $G(\theta)$. Therefore, to obtain $\mathcal{R}(\theta)$, we do not need to explore the collection of all possible information structures, but instead, we can calculate the set of 1BCE of $G(\theta)$. In the remainder of the section, we formalise the equivalent characterization of $\mathcal{R}(\theta)$ and Θ^* based on 1BCE. In Section 3.2, we zoom into the computational part. In Section 3.3, we discuss the identification power of our model. In Section 3.4, we characterize bounds on counterfactuals of interest.

First, we give the definition of 1BCE of $G(\theta)$. In what follows, $P_{Y,V}$ denotes the joint distribution of Y and V .

Definition 1. (1BCE of $G(\theta)$) Given $\theta \in \Theta$, $P_{Y,V}$ is a 1BCE of $G(\theta)$ if:

1. It is *consistent*, i.e., the marginal of $P_{Y,V}$ on Y is equal to the DM's prior, $P_V(\cdot; \theta_V)$:

$$\sum_{y \in \mathcal{Y}} P_{Y,V}(y, v) = P_V(v; \theta_V) \quad \forall v \in \mathcal{V}.$$

2. It is *obedient*, i.e., the DM who is recommended alternative $y \in \mathcal{Y}$ by an omniscient mediator

has no incentive to deviate:

$$\int_{\mathcal{V}} P_{Y,V}(y, v)(u(y, v; \theta_u) - u(y', v; \theta_u))dv \geq 0, \quad \forall y' \in \mathcal{Y} \setminus \{y\}, \forall y \in \mathcal{Y}.$$

We now state Theorem 1 of BM16.

Theorem 1. (BM16) Given $\theta \in \Theta$, $P_{Y,V}$ is a 1BCE of $G(\theta)$ if and only if there exists an information structure $S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S}$ and an optimal strategy $\mathcal{P}_{Y|T}$ of $\{G(\theta), S\}$ such that $P_{Y,V}$ arises from $\mathcal{P}_{Y|T}$.⁵ \diamond

We use Theorem 1 to equivalently rewrite $\mathcal{R}(\theta)$ and Θ^* . For each $\theta \in \Theta$, let $\mathcal{W}(\theta)$ be the set of 1BCEs of $G(\theta)$. Let $\mathcal{Q}(\theta)$ be the set of distributions of Y that arise from the 1BCEs of $G(\theta)$:

$$\mathcal{Q}(\theta) := \left\{ P_Y \in \Delta(\mathcal{Y}) : P_Y(y) = \int_{\mathcal{V}} P_{Y,V}(y, v)dv \quad \forall y \in \mathcal{Y}, P_{Y,V} \in \mathcal{W}(\theta) \right\}.$$

Theorem 1 implies that $\mathcal{R}(\theta) = \mathcal{Q}(\theta)$. This can be seen in three steps. First, observe that the *Consistency* and *Obedience* requirements defining 1BCE are linear in $P_{Y,V}$. Therefore, $\mathcal{Q}(\theta)$ is a convex set. Second, $\mathcal{R}(\theta)$ contains distributions of Y that are mixture of optimal strategies over various information structures. Hence, by Theorem 1, each $P_Y \in \mathcal{R}(\theta)$ maps into a mixture of 1BCEs. Third, since $\mathcal{Q}(\theta)$ is convex, any mixture of elements from the set is itself an element of $\mathcal{Q}(\theta)$. Therefore, $\mathcal{R}(\theta) = \mathcal{Q}(\theta)$ and we can use $\mathcal{Q}(\theta)$ in place of $\mathcal{R}(\theta)$ to characterize Θ^* . Proposition 1 follows.

Proposition 1. (*Equivalent characterization of Θ^* via 1BCE*) Let $\Theta^{**} := \{\theta \in \Theta : \mathbb{P}_Y \in \mathcal{Q}(\theta)\}$. It holds that $\Theta^* = \Theta^{**}$. \diamond

3.2 Construction of the identified set

To see how to use Proposition 1 in practice, we first rewrite it in a more explicit way using Definition 1. By Proposition 1 and Definition 1, θ belongs to Θ^* if and only if there exists a

⁵We say that $P_{Y,V}$ arises from $\mathcal{P}_{Y|T}$ if $P_{Y,V}(y, v) = \int_{t \in \mathcal{T}} P_{Y|T}(y|t)P_{T|V}(t|v)P_V(v; \theta_V)dt$, for every $y \in \mathcal{Y}$ and $v \in \mathcal{V}$.

function $P_{Y,V} : \mathcal{Y} \times \mathcal{V} \rightarrow \mathbb{R}$ which satisfies the following constraints:

$$\begin{aligned}
\text{Consistency:} \quad & \sum_{y \in \mathcal{Y}} P_{Y,V}(y, v) = P_V(v; \theta_V) \quad \forall v \in \mathcal{V}, \\
\text{Obedience:} \quad & \int_{\mathcal{V}} P_{Y,V}(y, v)(u(y, v; \theta_u) - u(y', v; \theta_u)) dv \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \\
\text{Probability requirements:} \quad & P_{Y,V}(y, v) \geq 0 \quad \forall y \in \mathcal{Y}, \forall v \in \mathcal{V}, \quad \sum_{y \in \mathcal{Y}} \int_{\mathcal{V}} P_{Y,V}(y, v) dv = 1, \\
\text{Data match:} \quad & \int_{\mathcal{V}} P_{Y,V}(y, v) dv = \mathbb{P}_Y(y) \quad \forall y \in \mathcal{Y}.
\end{aligned} \tag{2}$$

Observe that the above constraints are linear in $P_{Y,V}$. Therefore, when \mathcal{V} is a finite set, (2) reduces to a *finite-dimensional* linear program:

$$\begin{aligned}
\text{Consistency:} \quad & \sum_{y \in \mathcal{Y}} P_{Y,V}(y, v) = P_V(v; \theta_V) \quad \forall v \in \mathcal{V}, \\
\text{Obedience:} \quad & \sum_{v \in \mathcal{V}} P_{Y,V}(y, v)(u(y, v; \theta_u) - u(y', v; \theta_u)) \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \\
\text{Probability requirements:} \quad & P_{Y,V}(y, v) \geq 0 \quad \forall y \in \mathcal{Y}, \forall v \in \mathcal{V}, \quad \sum_{y \in \mathcal{Y}} \sum_{v \in \mathcal{V}} P_{Y,V}(y, v) = 1, \\
\text{Data match:} \quad & \sum_{v \in \mathcal{V}} P_{Y,V}(y, v) = \mathbb{P}_Y(y) \quad \forall y \in \mathcal{Y}.
\end{aligned} \tag{3}$$

In turn, we can construct Θ^* following this procedure: first, generate a grid of points covering Θ as precisely as possible, depending on the available computational resources; second, for each θ in such a grid, check if (3) has a solution with respect to $P_{Y,V}$; third, any θ for which (3) has a solution belongs to the identified set.⁶

Observe that, when \mathcal{V} is finite, we can dispense with the parameterisation of u and P_V via θ , as these functions can be fully and flexibly characterised by a finite number of parameters, one for each combination of values of (Y, V) . In this case, we can also add nonparametric restrictions on P_V to the linear program, such as monotonicity, concavity/convexity, and Lipschitz restriction, which can be written as linear constraints.

Further, we remark that (3) is linear in $P_{Y,V}$ for a given θ , but it is *not* linear in *both* $P_{Y,V}$ and θ . This is why we grid over Θ to construct Θ^* . Linearity of (3) in $P_{Y,V}$ and the parameters is

⁶Note that there is no need to recover the entire set of solutions of (3) for a given θ . Existence of at least one solution of (3) is sufficient to include such a θ in the identified set.

achieved when u is *known* and P_V is treated as a finite-dimensional vector of parameters without indexing it by θ_V . In this case, we can obtain the identified set of moments of P_V by solving a *unique* linear program, without gridding, as proposed by STZ21 for an auction framework. In single-agent decision problems and other types of games, u is typically unknown and, therefore, we have linearity *only* with respect to $P_{Y,V}$.

The setting with finite \mathcal{V} is considered by STZ21 and MR22. We refer to those papers for a discussion on the computational burden of solving (3) as the cardinalities of \mathcal{Y} and \mathcal{V} increase.

When \mathcal{V} is not a finite set, the simple finite-dimensional linear programming approach is no longer applicable as $P_{Y,V}$ is an *infinite-dimensional* object. This case has not been addressed in the econometric literature on games, where assuming a finite \mathcal{V} or discretising a non-finite \mathcal{V} in a few arbitrary bins to operationalise (2) are standard practices. Here we propose a formal procedure to approximate Θ^* when \mathcal{V} is not finite, along the lines of Han and Yang (2022). As a preliminary step, it is useful to equivalently rewrite (2) using the distribution of Y conditional on V as unknown, $\mathcal{P}_{Y|V} := \{P_{Y|V}(\cdot|v) : v \in \mathcal{V}\}$:⁷

$$\begin{aligned}
\text{Obedience:} & \quad \int_{\mathcal{V}} P_{Y|V}(y|v) P_V(v; \theta_V) (u(y, v; \theta_u) - u(y', v; \theta_u)) dv \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \\
\text{Probability requirements:} & \quad P_{Y|V}(y|v) \geq 0 \quad \forall y \in \mathcal{Y}, \forall v \in \mathcal{V}, \quad \sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) = 1 \quad \forall v \in \mathcal{V}, \\
\text{Data match:} & \quad \int_{\mathcal{V}} P_{Y|V}(y|v) P_V(v; \theta_V) dv = \mathbb{P}_Y(y) \quad \forall y \in \mathcal{Y}.
\end{aligned} \tag{4}$$

We start from the simple case where V is a random variable and $\mathcal{V} := [v_\ell, v_u]$. Consider the following sieve approximation of $P_{Y|V}(y|v)$ using Bernstein polynomials of order K :

$$P_{Y|V}(y|v) \approx \sum_{k \in \mathcal{K}} \lambda_{k,K}^y a_{k,K}(v), \tag{5}$$

where $a_{k,K}(v) := \binom{K}{k} \frac{(v-v_\ell)^k (v_u-v)^{K-k}}{(v_u-v_\ell)^K}$ is a univariate Bernstein basis, $\lambda_{k,K}^y := P_{Y|V}(y|v_\ell + (v_u-v_\ell) \frac{k}{K})$ is its coefficient, $\mathcal{K} := \{0, 1, \dots, K\}$, and K is finite. It can be shown that this approximation tends to $P_{Y|V}(y|v)$ uniformly over \mathcal{V} as K goes to infinity. Observe that $a_{k,K}(v) \geq 0$. Hence, for each y , $P_{Y|V}(y|v) \geq 0$ for each v if and only if $\lambda_{k,K}^y \geq 0$ for each k . Further, $\sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) = 1$

⁷Note that the *Consistency* constraint is redundant in (4) as $\sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) P_V(v; \theta_V) = P_V(v; \theta_V)$ becomes $\sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) = 1$, which is one of the probability requirements.

is approximately equal to $\sum_{y \in \mathcal{Y}} \lambda_{k,K}^y = 1$ for each k (Coolidge, 1949). Motivated by this result, a finite-dimensional linear program approximating (4) can be obtained with respect to $\lambda := (\lambda_{k,K}^y : (k, y) \in \mathcal{K} \times \mathcal{Y})$:

$$\begin{aligned}
\text{Obedience:} \quad & \sum_{k \in \mathcal{K}} \lambda_{k,K}^y \gamma_{1,k,K}^{y,y'}(\theta) \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \\
\text{Probability requirements:} \quad & \lambda_{k,K}^y \geq 0 \quad \forall k \in \mathcal{K}, \forall y \in \mathcal{Y}, \quad \sum_{y \in \mathcal{Y}} \lambda_{k,K}^y = 1 \quad \forall k \in \mathcal{K}, \\
\text{Data match:} \quad & \sum_{k \in \mathcal{K}} \lambda_{k,K}^y \gamma_{2,k,K}(\theta_V) = \mathbb{P}_Y(y) \quad \forall y \in \mathcal{Y},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\gamma_{1,k,K}^{y,y'}(\theta) &:= \int_{\mathcal{V}} a_{k,K}(v) P_V(v; \theta_V) (u(y, v; \theta_u) - u(y', v; \theta_u)) dv, \\
\gamma_{2,k,K}(\theta_V) &:= \int_{\mathcal{V}} a_{k,K}(v) P_V(v; \theta_V) dv.
\end{aligned}$$

We can then approximate Θ^* by verifying if (6) has a solution with respect to λ for each $\theta \in \Theta$.

The same logic applies when V is a $D \times 1$ random vector and $\mathcal{V} := \times_{d=1}^D [v_{\ell_d}, v_{u_d}]$. In this case, we can use (5), where $\mathcal{K} := \times_{d=1}^D \{0, 1, \dots, K_d\}$, K_d is finite for each $d = 1, \dots, D$, K is the cardinality of \mathcal{K} , $a_{k,K}(v) := \prod_{d=1}^D \binom{K_d}{k_d} \frac{(v_d - v_{\ell_d})^{k_d} (v_{u_d} - v_d)^{K_d - k_d}}{(v_{u_d} - v_{\ell_d})^{K_d}}$ is a D -variate Bernstein basis, and $\lambda_{k,K}^y := P_{Y|V}(y | v_{\ell_1} + (v_{u_1} - v_{\ell_1}) \frac{k_1}{K_1}, \dots, v_{\ell_D} + (v_{u_D} - v_{\ell_D}) \frac{k_D}{K_D})$ is its coefficient. We can further generalise Bernstein polynomials to the case where \mathcal{V} is the real line (Szasz, 1950; Butzer, 1954).

Instead of applying the above procedure, we could approximate (4) using an i.i.d. sample, $\{V_j\}_{j=1}^J$, from $P_V(\cdot; \theta_V)$ (Calafiore and Campi, 2005). This method is similar to the discretisation in bins by MR22. However, as discussed by Han and Xu (2022) in their Remark 6.1, the linear program (6) may be more stable.

Suppose there are other *discrete* payoff-relevant variables which enter the DM's information set together with the signal. In this case, one should solve (6) for each value of such variables. For instance, u could depend on covariates X observed by the DM and the researcher. u could also depend on some variable ϵ observed by the DM but unobserved by the researcher. Our procedure allows (X, V, ϵ) to be correlated, so that DMs can have heterogenous priors, $P_{V|X,\epsilon}(\cdot; |x, e; \theta_V)$ conditional on $(X, \epsilon) = (x, e)$. See Appendix C on how to include (X, ϵ) in (6).

Verifying if (6) has a solution for a given θ is computationally easy using standard algorithms, such as the simplex optimizer or the interior-point optimizer. In Section 4, we discuss how to

choose K in practice and the computational time.

3.3 On the identification power of the model

Our identification procedure imposes weak assumptions on the DMs' behaviour by allowing for any information structures via the 1BCE characterization. Proposition 2 explores whether such a "light" framework retains identification power with respect to (u, P_V) without relying on variation from exogenous covariates and (non)parametric restrictions.

Proposition 2. (*Identification power*) (i) For every \mathcal{Y} and \mathcal{V} , there exists a choice distribution, $P_Y \in \Delta(\mathcal{Y})$, such that, when $\mathbb{P}_Y := P_Y$, the identified set $\Theta^* := \{(u, P_V) \in \mathcal{U} \times \Delta(\mathcal{V}) : \mathbb{P}_Y \in \mathcal{R}(u, P_V)\}$ is a strict subset of $\mathcal{U} \times \Delta(\mathcal{V})$.

(ii) Let $V := (V_y : y \in \mathcal{Y})$ be a $|\mathcal{Y}| \times 1$ vector whose distribution P_V has full support and is absolutely continuous with respect to the Lebesgue measure. Let $\tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ be the set of all such P_V . Let $u(y, v) = \tilde{u}(y) + v_y$ for each $(y, v) \in \mathcal{Y} \times \mathcal{V}$, where $\tilde{u} : \mathcal{Y} \rightarrow \mathbb{R}$. Let $\tilde{\mathcal{U}}$ be the set of all such \tilde{u} . Let $\Theta^* := \{(\tilde{u}, P_V) \in \tilde{\mathcal{U}} \times \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|}) : \mathbb{P}_Y \in \mathcal{R}(\tilde{u}, P_V)\}$ be the identified set under these restrictions. Then, for every \mathcal{Y} : (a) the projection of Θ^* on $\tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ is equal to $\tilde{\mathcal{U}}$; (b) the projection of Θ^* on $\tilde{\mathcal{U}}$ is equal to $\tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$. \diamond

Part (i) of Proposition 2 shows that Θ^* can be a strict subset of the parameter space, $\mathcal{U} \times \Delta(\mathcal{V})$. Therefore, even in the absence of exogenous covariates and (non)parametric restrictions, our framework can have sufficient identification power to exclude some pairs (u, P_V) from the identified set. Part (ii) of Proposition 2 digs deeper by investigating the projections of the identified set, under continuity of V and a nonparametric restriction on the utility function that is common in the literature on discrete choice models, namely additive separability between the systematic payoff, \tilde{u} , and the idiosyncratic state of the world, V . It shows that such projections include all \tilde{u} and all P_V . Possible solutions to turn the negative identification result of part (ii) into a positive one consist of allowing \tilde{u} to depend on covariates which are independent of V , assuming that (\tilde{u}, P_V) belong to parametric families, incorporating nonparametric restrictions on P_V , such as quantile restrictions, etc. In our simulations and empirical application, we show that the model can generate relatively tight bounds by introducing exogenous covariates and parameterising (\tilde{u}, P_V) .

One may wonder whether the negative identification result of part (ii) is caused by the agnosticism about information structures and the focus on 1-player games rather than many-player games. The proof of part (ii) reveals that this is not the case, as we now explain.

To show part (ii), we start under the assumption that all DMs process the complete information structure so that our framework becomes a standard additive random utility discrete choice model. We then apply two well-known results in the literature on identifying additive random utility discrete choice models. These are Theorem 1 of Haile, Hortaçsu, and Kosenok (2008) and Proposition 1 of Galichon and Salanié (2022).

Haile, Hortaçsu, and Kosenok (2008) investigate the identification power of the notion of quantile response equilibrium (QRE) in N -player games with $N \geq 1$ and additively separable shocks. In their Theorem 1, they show that QRE imposes no restrictions on behaviour. That is, for *any* systematic payoff and distribution of observed behaviour, there exists a density of the additively separable unobserved terms entering payoffs such that a QRE generates the distribution of observed behaviour. When $N = 1$, the QRE framework reduces to an additive random utility discrete choice model (Footnote 9, p.184, Haile, Hortaçsu, and Kosenok, 2008). Hence, we can apply Theorem 1 of Haile, Hortaçsu, and Kosenok (2008) to our model and show that, when all DMs process the complete information structure, any \tilde{u} rationalises the empirical choice distribution. Clearly, such a result remains valid when we stay agnostic about information structures and part (ii)-(a) follows.

Galichon and Salanié (2022) study identification in 2-sided matching models with additively separable shocks. In their Proposition 1, they show that 2-sided matching models are under-identified. Specifically, for *any* density of the additively separable unobserved terms entering payoffs and distribution of observed behaviour, there exists a systematic payoff such that the model-implied stable match generates the distribution of observed behaviour. The result also applies to the 1-sided case of an additive random utility discrete choice model. Hence, we can apply Proposition 1 of Galichon and Salanié (2022) to our model and show that, when all DMs process the complete information structure, any P_V rationalises the empirical choice distribution. Clearly, such a result remains true when we stay agnostic about information structures and part (ii)-(b) follows.

This proof highlights that the negative identification result of part (ii) is not caused by the agnosticism about information structures and the focus on 1-player games. In fact, parts (ii)-(a) and (ii)-(b) hold even under the assumption that all DMs process the complete information structure, as discussed above. Further, part (ii)-(a) also holds in many-player games. Specifically, when $N \geq 2$, the QRE framework can be reinterpreted as a game with incomplete information and Bayesian Nash equilibrium (Footnote 4, p.182, Haile, Hortaçsu, and Kosenok, 2008). In that setting, Theorem 1 of Haile, Hortaçsu, and Kosenok (2008) shows that, for any systematic payoff and distribution of observed behaviour, there exists a density of the additively separable unobserved terms entering payoffs such that a Bayesian Nash equilibrium generates the distribution of observed behaviour. Again, such a result remains true when we stay agnostic about information structures and part (ii)-(a) follows. To sum up, our framework should be enriched by other assumptions on (\tilde{u}, P_V) , over and above additive separability, to generate identification power like in standard additive random utility discrete choice models and several many-player games.

Nevertheless, even in the presence of covariates and parametric restrictions, we expect our framework to generally provide less informative bounds on the primitives than 2-player games and no assumptions on information structures. To see why, consider a 2-player entry game with $\mathcal{Y} := \{0, 1\}$. Let the payoff of player $i \in \{1, 2\}$ be $Y_i(X_i\beta + \delta Y_j + \epsilon_i)$, where Y_j is the competitor's action, X_i represents i 's covariates (scalar, for simplicity), and ϵ_i captures i 's characteristics unobserved by the researcher. Each player i is assumed to observe (X_i, X_j, ϵ_i) . The researcher remains agnostic about i 's knowledge of ϵ_j and, hence, their ability to accurately predict Y_j . Note that this framework resembles our model (game against nature) by letting player i be the DM, assigning player j 's role to nature, setting $Y_j := V$, and pretending that the researcher observes V . However, as discussed in Section 2, there is a key difference that makes our framework non-nested with the above 2-player game: while in the above 2-player game player j chooses Y_j based on best-response mechanisms, in our setting nature is indifferent among outcomes and selects V randomly from P_V . This has implications for identification. In particular, a common way to show *point* identification of β and the parameters governing the distribution of (ϵ_1, ϵ_2) in the above 2-player game is to use “at infinity” arguments (Theorem 1, Tamer, 2003; Proposition 3, MR22). These consist of finding extreme values of X_j that induce player j always to choose one

action, so that player i 's problem turns to a single agent parametric discrete choice problem with complete information that we know to be point identified (Manski, 1988). Such a strategy is not implementable in our setting because the model lacks assumptions regulating nature's behaviour, let alone covariates influencing the choice of V by nature. Hence, point identification is likely to fail even in the presence of special regressors, which hints at a generic reduction in the identification power of our framework with respect to 2-player games.

3.4 Counterfactual identification

A key objective in the empirical analysis of single-agent decision problems is predicting how individual choices and social welfare change in counterfactual decision settings deployed in the same environment. Our framework allows us to consider two types of policy experiments, illustrated in what follows.

3.4.1 Information provision

In the first policy experiment, we study how the choice probabilities change in response to changes in the availability of information to agents about the state of the world. Suppose the policy maker implements some intervention that urges all agents to process information structure S . Some agents may settle for this information structure, while others might prefer to collect *further* information. In this new environment, the DM selects their favourite alternative from \mathcal{Y} by solving the augmented decision problem $\{G(\theta_0), S^\dagger\}$, where S^\dagger is an unknown information structure that is *at least as* informative as S . S^\dagger is also called an *expansion* of S . Formally, $S^\dagger := \{\mathcal{T}^\dagger, \mathcal{P}_{\mathcal{T}^\dagger|V}\}$ is an expansion of $S := \{\mathcal{T}, \mathcal{P}_{\mathcal{T}|V}\}$ if there exists $S^\diamond := \{\mathcal{T}^\diamond, \mathcal{P}_{\mathcal{T}^\diamond|V}\}$ such that S^\dagger is the *combination* of S and S^\diamond . That is, $\mathcal{T}^\dagger := \mathcal{T} \times \mathcal{T}^\diamond$, $\int_{\mathcal{T}^\diamond} P_{\mathcal{T}^\dagger|V}(t, t^\diamond|v) dt^\diamond = P_{\mathcal{T}|V}(t|v)$ for each $t \in \mathcal{T}$ and $v \in \mathcal{V}$, and $\int_{\mathcal{T}} P_{\mathcal{T}^\dagger|V}(t, t^\diamond|v) dt = P_{\mathcal{T}^\diamond|V}(t^\diamond|v)$ for each $t^\diamond \in \mathcal{T}^\diamond$ and $v \in \mathcal{V}$ (Definition 5, BM16).

We can characterize sharp bounds on the counterfactual choice probabilities while fixing the policy-implemented information structure S and remaining agnostic about S^\diamond and, hence, S^\dagger . To do this, we use once again Theorem 1 of BM16, which generically applies to any 1-player game whose *minimal* information structure S has been set by the researcher. Namely, by Theorem 1 of BM16, the set of optimal strategies that arise from arbitrarily expanding S is equivalent to

the set of 1BCE of $\{G(\theta_0), S\}$. Therefore, the collection of counterfactual choice probabilities is simply the set of 1BCE of $\{G(\theta_0), S\}$. Proposition 3 formalises these arguments.

Proposition 3. (*Counterfactual bounds - information provision*) Let $S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S}$ be the information structure implemented by a policy program. Let $\mathcal{P}_{Y^\dagger}^*(h)$ be the identified set of the counterfactual probability of choosing alternative $h \in \mathcal{Y}$ when DMs face the augmented decision problem $\{G(\theta_0), S^\dagger\}$, where $S^\dagger := \{\mathcal{T}^\dagger, \mathcal{P}_{T^\dagger|V}\} \in \mathcal{S}$ is some unknown expansion of S , potentially heterogenous across DMs. Then, $\mathcal{P}_{Y^\dagger}^*(h) = \cup_{\theta \in \Theta^*} [\Psi^\ell(h; \theta), \Psi^u(h; \theta)]$, where

$$\Psi^\ell(h; \theta) := \min_{P_{Y^\dagger, V, T}: \mathcal{Y} \times \mathcal{V} \times \mathcal{T} \rightarrow \mathbb{R}} \int_{\mathcal{V}} \int_{\mathcal{T}} P_{Y^\dagger, V, T}(h, v, t) dt dv \text{ s.t. (7) - (9),}$$

$$\Psi^u(h; \theta) := \max_{P_{Y^\dagger, V, T}: \mathcal{Y} \times \mathcal{V} \times \mathcal{T} \rightarrow \mathbb{R}} \int_{\mathcal{V}} \int_{\mathcal{T}} P_{Y^\dagger, V, T}(h, v, t) dt dv \text{ s.t. (7) - (9),}$$

and

$$\sum_{y^\dagger \in \mathcal{Y}} P_{Y^\dagger, V, T}(y^\dagger, v, t) = P_V(v; \theta_V) P_{T|V}(t|v) \quad \forall v \in \mathcal{V}, \forall t \in \mathcal{T}, \quad (7)$$

$$\int_{\mathcal{V}} P_{Y^\dagger, V, T}(y^\dagger, v, t) (u(y^\dagger, v; \theta_u) - u(k^\dagger, v; \theta_u)) dv \geq 0 \quad \forall y^\dagger \in \mathcal{Y}, \forall k \in \mathcal{Y} \setminus \{y^\dagger\}, \forall t \in \mathcal{T}, \quad (8)$$

$$P_{Y^\dagger, V, T}(y^\dagger, v, t) \geq 0 \quad \forall y^\dagger \in \mathcal{Y}, \forall v \in \mathcal{V}, \forall t \in \mathcal{T}, \quad \sum_{y^\dagger \in \mathcal{Y}} \int_{\mathcal{V}} \int_{\mathcal{T}} P_{Y^\dagger, V, T}(y^\dagger, v, t) dt dv = 1. \quad (9)$$

◇

In Proposition 3, (7)-(9) define a 1BCE of $\{G(\theta), S\}$. In particular, (7) is the *Consistency* constraint, (8) is the *Obedience* constraint, and (9) ensures that $P_{Y^\dagger, V, T}$ is a proper distribution. Observe that computing $\Psi^\ell(h; \theta)$ and $\Psi^u(h; \theta)$ for a given θ requires us to solve two linear programs with respect to $P_{Y^\dagger, V, T}$. Further, Proposition 3 presumes that the identified set, Θ^* , has been constructed in a pre-step. One could also append the program of Proposition 3 to (2), solve a unique program, and thus find $\mathcal{P}_{Y^\dagger}^*(h) \times \Theta^*$ in one step. However, this strategy would not bring notable computational advantages because the programs of Proposition 3 and (2) are not linear in θ .

Understanding how information changes behaviour is a key question in the literature on single-agent decision problems (see, for example, Athey, 2002) and is largely absent in the literature on

many-player games. In particular, as discussed in Section 1, several empirical papers are concerned with assessing the impact on choices of sending agents information about payoff-relevant variables. These papers typically exploit field experiments which can be very costly. Proposition 3 represents a powerful result because it allows the analyst to assess the effect of programs of information provision *prior* to conducting such interventions. In the empirical application of Section 5, we apply Proposition 3 to the benchmark case where the policy intervention fully reveals the state of the world to agents, i.e., S is the complete information structure and, therefore, $S^\dagger = S$ for each DM.⁸

The complete information benchmark is also useful to evaluate the welfare cost of limited information which may keep agents from choosing their first best. Specifically, given the true parameter vector $\theta_0 \in \Theta$, we consider the average gains in the model-implied *ex-post* utility from expanding every DM's information structure from the null information structure to the complete information structure:

$$\begin{aligned} \Delta \mathbb{E}_{\theta_0} := & \int_{\mathcal{V}} u\left(\operatorname{argmax}_{y \in \mathcal{Y}} u(y, v; \theta_{0,u}), v\right) P_V(v; \theta_{0,V}) dv \\ & - \int_{\mathcal{V}} u\left(\operatorname{argmax}_{y \in \mathcal{Y}} \int_{\mathcal{V}} u(y, v; \theta_{0,u}) P_V(v; \theta_{0,V}) dv, v\right) P_V(v; \theta_{0,V}) dv, \end{aligned} \tag{10}$$

where the first term is the average ex-post utility when all agents process the complete information structure and the second term is the average ex-post utility when all agents process the null information structure. Observe that, in the complete information scenario, each DM earns an ex-post payoff that is, on average, greater or equal than the ex-post payoff under the null information structure. Therefore, $\Delta \mathbb{E}_{\theta_0}$ captures the maximum welfare cost of limited information. The identified set of $\Delta \mathbb{E}_{\theta_0}$ is

$$W^* := \cup_{\theta \in \Theta^*} \Delta \mathbb{E}_{\theta}.$$

3.4.2 Changes in covariates

The second policy experiment considers a more traditional, yet important, question that interests the literature on both single-agent decision problems and games. In particular, we study how the

⁸Any expansion S^\dagger of the complete information structure S is equal to S .

choice probabilities change in response to changes in covariates X entering the utility function and observed by the researcher and the DMs. Suppose the policy maker implements some intervention which shifts the realization of X assigned to agents. In this new environment, the DM processes the same information structure as in the factual scenario (i.e., differently from Proposition 3, information structures are now held fixed), but has to account for the new realization of X in evaluating their payoffs. Proposition 4 characterises sharp bounds on the counterfactual choice probabilities.

Before presenting Proposition 4, we introduce some useful notation and provide the intuition behind the result. Let \mathcal{X} be the finite support of X . Let X^\dagger be the vector of covariates after the intervention, with finite support \mathcal{X}^\dagger . To compute the counterfactual choice probabilities, imagine a hypothetical scenario where the DM simultaneously chooses an alternative from \mathcal{Y} for the augmented decision problem $\{G(\theta_0, X), S\}$ and an alternative from \mathcal{Y} for the augmented decision problem $\{G(\theta_0, X^\dagger), S\}$. The choices for $\{G(\theta_0, X), S\}$ and $\{G(\theta_0, X^\dagger), S\}$ are denoted by Y and Y^\dagger , respectively. Y^\dagger and S are not observed by the researcher. There is no interaction between the two choices except for the common information structure. $\{G(\theta_0, X), G(\theta_0, X^\dagger), S\}$ is called the *linked augmented decision problem*. $\{G(\theta_0, X), G(\theta_0, X^\dagger)\}$ is called the *linked baseline decision problem*. Let $\mathcal{P}_{Y, Y^\dagger, V|X, X^\dagger}$ be a 1BCE of the linked baseline decision problem and consider the set of such 1BCEs. By Theorem 1 above, the marginal of these distributions on (Y^\dagger, V) is precisely the set of counterfactual choice probabilities that we are looking for. These steps are formalised by Theorem 1 in Bergemann, Brooks, and Morris (2022) and readapted to our case by Proposition 4.

Proposition 4. (*Counterfactual bounds - change in covariates*) Let $\mathcal{Q}_{Y^\dagger|x, x^\dagger}^*(h)$ denote the identified set of the counterfactual probability of choosing alternative $h \in \mathcal{Y}$ when DMs face the augmented decision problem $\{G(\theta_0, x^\dagger), S\}$, where S is the unknown information structure processed in the factual scenario and potentially heterogenous across DMs, and $x \in \mathcal{X}$ and $x^\dagger \in \mathcal{X}^\dagger$ are the covariate realizations before and after the policy intervention, respectively. Then,

$\mathcal{Q}_{Y^\dagger|x,x^\dagger}^*(h) = \cup_{\theta \in \Theta^*} [\Phi^\ell(h|x, x^\dagger; \theta), \Phi^u(h|x, x^\dagger; \theta)]$, where

$$\Phi^\ell(h|x, x^\dagger; \theta) := \min_{P_{Y,Y^\dagger,V|X,X^\dagger}(\cdot|x,x^\dagger): \mathcal{Y}^2 \times \mathcal{V} \rightarrow \mathbb{R}} \sum_{y \in \mathcal{Y}} \int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, h, v|x, x^\dagger) dv. \text{ s.t. (11) - (15),}$$

$$\Phi^u(h|x, x^\dagger; \theta) := \max_{P_{Y,Y^\dagger,V|X,X^\dagger}(\cdot|x,x^\dagger): \mathcal{Y}^2 \times \mathcal{V} \rightarrow \mathbb{R}} \sum_{y \in \mathcal{Y}} \int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, h, v|x, x^\dagger) dv. \text{ s.t. (11) - (15),}$$

and

$$\sum_{y \in \mathcal{Y}} \sum_{y^\dagger \in \mathcal{Y}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) = P_V(v|x, x^\dagger; \theta_{0,V}) \quad \forall v \in \mathcal{V}, \quad (11)$$

$$\int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) (u(y, v, x; \theta_{0,u}) - u(y', v, x; \theta_{0,u})) dv \geq 0, \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \forall y^\dagger \in \mathcal{Y}, \quad (12)$$

$$\int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) (u(y^\dagger, v, x^\dagger; \theta_{0,u}) - u(k^\dagger, v, x^\dagger; \theta_{0,u})) dv \geq 0, \quad \forall y^\dagger \in \mathcal{Y}, \forall k^\dagger \in \mathcal{Y} \setminus \{y^\dagger\}, \forall y \in \mathcal{Y}, \quad (13)$$

$$P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) \geq 0 \quad \forall y \in \mathcal{Y}, \forall y^\dagger \in \mathcal{Y}, \forall v \in \mathcal{V}, \sum_{y \in \mathcal{Y}} \sum_{y^\dagger \in \mathcal{Y}} \int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) dv = 1, \quad (14)$$

$$\sum_{y^\dagger \in \mathcal{Y}} \int_{\mathcal{V}} P_{Y,Y^\dagger,V|X,X^\dagger}(y, y^\dagger, v|x, x^\dagger) dv = \mathbb{P}_Y(y|x) \quad \forall y \in \mathcal{Y}. \quad (15)$$

◇

In proposition 4, (11)-(15) define a 1BCE of the linked baseline decision problem $\{G(\theta, X), G(\theta, X^\dagger)\}$. In particular, (11) is the *Consistency* constraint, (12) and (13) are the *Obedience* constraints for each decision problem, (14) ensures that $P_{Y,Y^\dagger,V|X,X^\dagger}$ is a proper distribution, and (15) imposes that the factual choice probabilities are equal to the empirical ones.

In the empirical application of Section 5, we apply Proposition 4 to study how the well-being of UK parties changes when they modify their ideological positions on popular policy issues.

4 Simulations

We consider a model specification close to the one used in the empirical application of Section 5. The payoff function is

$$u(y, X_i, V_i; \beta) := \beta X_{iy} + V_{iy}, \quad (16)$$

where i indexes a generic DM, $\mathcal{Y} := \{0, 1, \dots, D\}$, 0 denotes the outside option and its utility is normalised to zero, X_{iy} and V_{iy} are DM-alternative specific features. DM i observes the realization of $X_i := (X_{i1}, \dots, X_{iD})$ but may be uncertain about the realization of $V_i := (V_{i1}, \dots, V_{iD})$. X_i and

V_i are assumed to be independent. DM i has a prior on V_i , which is assumed to be standard Normal. The researcher observes the choice made by DM i and the realization of X_i for a large sample of DMs, without knowing their information structures. Note that this framework reduces to a standard multinomial Probit model under the additional assumption that each DM processes the complete information structure.

In the first simulation exercise, we illustrate how to choose the order of the Bernstein polynomials, K . As developing data-driven procedures to choose K is still an open question in nonparametric frameworks with point identification, here we follow the heuristic approach developed by Han and Yang (2022) for partially identified settings. We simulate the data from (16) with $|\mathcal{Y}| = 3$, $D = 2$, and $\beta = 1.3$. Each X_{iy} is randomly drawn from a probability mass function constructed by taking a bivariate normal and then discretising it to have support $\mathcal{X} := \{-2.4, -0.4, 0.3\}$.⁹ The data is generated assuming all DMs process the complete information structure. The first column of Table 1 reports the identified set as K increases. To obtain such identified set, we explored a grid of candidate values of β between -30 and 30 equally distanced at 0.001. The second and third columns report K_d , which is taken to be constant across $d = 1, \dots, D$, and $K := (K_d + 1)^D$, respectively. For a given covariate realization x and parameter value β , the fourth column computes the number of unknowns of the linear program (6), which is $(D + 1)K$. Observe that the width of the bounds tends to increase weakly with K . When $K_d = 3$ ($K = 64$), the bounds are narrow, but this may be due to our misspecification of the smoothness of the family of functions to be approximated, $\mathcal{P}_{Y|V}$. When K_d is sufficiently large, the width increases at a slower rate, and the bounds start to converge. In particular, the bounds become stable from $K_d = 10$ ($K = 1,331$) onwards. Therefore, we set $K_d = 10$ for $d = 1, \dots, D$ in the next simulations.¹⁰

The fifth column of Table 1 shows the average CPU time to assess if (6) has a solution for a given (x, β) , using the MOSEK solver for Matlab. The CPU time includes the calculation

⁹That is, $\Pr(X = x) = \frac{\exp(-(x-\mu)^2/\sigma^2)}{\sum_{x \in \mathcal{X}} \exp(-(x-\mu)^2/\sigma^2)}$ for each $x \in \mathcal{X}$.

¹⁰When the data are generated from (16) with X_i exogenous and V_i distributed as a standard Normal, then $\beta = 0$ is expected to be part of the identified set. To see why, suppose $\beta = 0$ and each DM processes the null information structure, so that the posterior is equal to the prior. Then, given the standard Normal prior, each DM gets the same expected payoff, $\mathbb{E}(V_{iy}) = 0$, from every alternative $y \in \mathcal{Y}$ and chooses according to some tie-breaking rule. Clearly, there will be a tie-breaking rule that reproduces the data. Hence, $\beta = 0$, coupled with the null information structure for all DMs and some tie-breaking rule, generates the data. Nevertheless, the model maintains enough identification power to exclude negative values of β , as shown in Table 1.

of the integral $\gamma_{1,k,K}^{y,y',x}(\beta) := \int_{\mathcal{Y}} a_{k,K}(v)P_V(v)(u(y,x,v;\beta) - u(y',x,v;\beta))dv$ for each $k \in \mathcal{K} := \times_{d=1}^D \{0, 1, \dots, K_d\}$ and $y, y' \in \mathcal{Y}$.¹¹ These integrals are computed by Monte Carlo integration taking 10^4 random draws from the standard Normal distribution. The number of unknowns of (6) increases linearly in K and exponentially in K_d . Hence, the CPU time increases approximately linearly in K and exponentially in K_d . The total time to construct the identified set depends on the possibility of parallelising across (x, β) . Using a computing cluster, we could exploit 600 parallel workers by coding our procedure as an SGE array job. Based on those, the rough total CPU time is listed in the last column of Table 1.

Identified set	$K_d, d = 1, \dots, D$	Order polynomial $K := (K_d + 1)^D$	Unknowns in (6) $(D + 1)K$	CPU time per (x, β)	Total CPU time
[0, 1.321]	3	16	48	0.046 s	42 s
[0, 1.402]	5	36	108	0.128 s	115 s
[0, 1.504]	7	64	192	0.226 s	204 s
[0, 1.565]	10	121	363	0.463 s	7 min
[0, 1.565]	15	256	7688	0.923 s	14 min
[0, 1.565]	20	441	1,323	1.539 s	23 min
[0, 1.565]	25	676	2,028	2.762 s	41 min
[0, 1.565]	30	961	2,883	3.561 s	53 min

Note: The true value of β is 1.3. $D = 2$. The CPU time per (x, β) includes the calculation of $\{\gamma_{1,k,K}^{y,y',x}(\beta)\}$ using 10^4 random draws from the standard Normal distribution. The total CPU time is based on 600 parallel workers, a grid of candidate values of β between -30 and 30 equally distanced at 0.001, and 9 possible realizations of X_i .

Table 1: Identified set in the first simulation exercise.

In the second simulation exercise, we investigate how the identified set varies as the cardinality of the support of X_i increases. We generate data from (16) with $|\mathcal{Y}| = 3$, $D = 2$, $\beta = 1.3$, and all DMs processing the complete information structure. We distinguish two scenarios regarding the dependence between X_i and V_i . In the first scenario, X_i and V_i are independent and the prior on V_i is a standard normal, as in the first simulation exercise. In the second scenario, X_i and V_i are allowed to be correlated and the prior on V_i conditional on $X_i = x$ is a normal distribution with mean and variance that vary with x . For each scenario, we study three cases. In the first case, each X_{iy} is randomly drawn from a probability mass function constructed by taking a bivariate normal and then discretising it to have support $\mathcal{X} := \{-2.4, -0.4, 0.3\}$, as in the first simulation exercise. In the second case, $\mathcal{X} := \{-2.4, -0.5, -0.4, 0.1, 0.3\}$. In the third case, $\mathcal{X} := \{-2.4, -0.7, -0.5, -0.4, 0.1, 0.2, 0.3\}$. We set $K_d = 10$ for $d = 1, \dots, D$ to construct

¹¹Recall that P_V is assumed to be standard normal, with V_i independent of X_i . Hence, $\gamma_{2,k,K}^{y,y'} := \int_{\mathcal{Y}} a_{k,K}(v)P_v(v)dv$ does not vary across values of the covariates and parameters and can be computed once.

the identified set. For each \mathcal{X} , the second and third columns of Table 2 show the identified set in the first and second scenarios, respectively. When X_i is exogenous, we find that the identified set shrinks as the cardinality of \mathcal{X} increases. This is because variation in exogenous covariates induces variation in agents' choices, which helps the identification of β , as in standard parametric analysis. Conversely, when X_i and V_i are allowed to be correlated, the identified set becomes larger as the cardinality of \mathcal{X} increases. This is because, as the cardinality of \mathcal{X} increases, there are smaller groups of DMs with the same prior, which worsens the identification of β by “weakening” the *Consistency* requirement of 1BCE.¹²

Support of X_{iy}	Identified set (X_i, V_i) independent	Identified set (X_i, V_i) correlated
$\mathcal{X} := \{-2.4, -0.4, 0.3\}$	[0, 1.565]	[0.548, 1.317]
$\mathcal{X} := \{-2.4, -0.5, -0.4, 0.1, 0.3\}$	[0, 1.505]	[0.494, 1.448]
$\mathcal{X} := \{-2.4, -0.7, -0.5, -0.4, 0.1, 0.2, 0.3\}$	[0, 1.474]	[0.479, 1.627]

Note: The true value of β is 1.3.

Table 2: Identified set in the second simulation exercise.

In the third simulation exercise, we investigate how the identified set changes as the information structures processed by DMs in the underlying data generating process vary. We generate data from (16) with $|\mathcal{Y}| = 3$, $D = 2$, $\beta = 1.3$, X_i independent of V_i , each X_{iy} randomly drawn from a probability mass function constructed by taking a bivariate normal and then discretising it to have support $\mathcal{X} := \{-2.4, -0.4, 0.3\}$, and V_i distributed as a standard Normal. We set $K_d = 10$ for $d = 1, 2$ to construct the identified set. We consider three scenarios. In the first scenario, all DMs process the complete information structure, as in the first simulation exercise. In the second scenario, 5/6 of the DMs process the complete information structure, and 1/6 process the null information structure. In the third scenario, all DMs process the null information structure. For each scenario, Table 3 shows the identified set (second column) and the value of β that would be identified if the researcher assumed that all DMs process the complete information structure (third

¹²When the data are generated from (16) and the prior varies across realizations of X_i , $\beta = 0$ may not be part of the identified set. To see why the logic of Footnote 10 does not apply, suppose $\beta = 0$ and each DM processes the null information structure, so that the posterior is equal to the prior. If $X_i = x$, then DM i gets an expected payoff equal to $\mathbb{E}(V_{iy}|X_i = x) := \mu_{y,x}$ from every alternative $y \in \mathcal{Y} \setminus \{0\}$, and a payoff equal to 0 from the outside option. Hence, the DMs' choices will differ depending on the realization of the covariates and the tie-breaking rule adopted. In turn, the resulting distribution of choices may not coincide with the empirical one and, so, $\beta = 0$, may not belong to the identified set.

column). Assuming that all DMs process the complete information structure, as in a standard multinomial Probit model, leads to recovering one parameter value contained in the identified set. When this assumption is misspecified (second and third scenario), the recovered parameter value differs from the truth. The model has the least identifying power in the third scenario, when all DMs process the null information structure. As soon as a significant proportion of DMs process the complete information structure (first and second scenarios), the identifying power of the model improves. Note, in fact, that, under the first scenario, the DMs take their decisions based on the actual payoffs. Instead, under the third scenario, the DMs choose based on the expected payoffs which are relatively homogenous in the population because computed using the same posterior. Hence, under the third scenario, there is less variation in the DMs’ choices, which leads to wider bounds.

Information structure	Identified set	β^{com}
Complete	$[0, 1.565]$	1.3
Mixture	$[0, 1.706]$	1.632
Null	$[0, \infty)$	15.186

Note: The true value of β is 1.3.

Table 3: Identified set in the third simulation exercise.

5 Empirical application

In this section, we use our methodology to study the determinants of voting behaviour during the UK general election held on 8 June 2017 and perform some counterfactual exercises aiming to evaluate the impact of uncertainty on the well-being of voters and parties.

5.1 Setting and model specification

The spatial model of voting is a dominant framework in political economy to explain individual preferences for parties and, in turn, how such preferences shape the policies implemented by democratic societies (Downs, 1957; Black, 1958). This model posits that an agent has a most preferred policy (also called “bliss point”) and casts their vote in favour of the party whose position is closest to their ideal (i.e., she votes “ideologically”). In empirical analysis, it is typically

implemented by estimating a classical parametric discrete choice model with perfect information. That is, it is assumed that each DM i processes the complete information structure and votes for party $y \in \mathcal{Y}$ maximising their utility,

$$u(y, X_i, V_i; \theta_u) := \beta^\top Z_{iy} + \gamma_y^\top W_i + V_{iy},$$

where $Z_{iy} := |Z_i - Z_y|$ is an $M \times 1$ vector observed by the researcher, representing the distance between DM i 's opinion (Z_i) and party y 's opinion (Z_y) on M issues, as measured in some common M -dimensional ideological metric space. W_i is a vector of individual-specific covariates observed by the researcher. $X_i := (W_i, Z_{iy} : y \in \mathcal{Y})$ collects the ideological distances and the individual-specific covariates. $V_i := (V_{iy} : y \in \mathcal{Y})$ is a vector of tastes of DM i for each party/candidate that is unknown to the researcher and independent of X_i , whose distribution belongs to a parametric family, thereby outlining a (Multinomial) Logit model, Probit model, Nested Logit model, etc. If voters vote ideologically, then β is expected to be negative so that DM i 's utility declines with increasing distance between Z_i and Z_y . $\theta_u := (\beta, \gamma_y : y \in \mathcal{Y})$ is the vector of payoff parameters.

The above framework is scientifically appealing because of its elegance and simplicity but it has limitations. Importantly, uncertainty affects voting (Matějka and Tabellini, 2021, and other references in Section 1). That is, voters may be unsure about their own and the parties' ideological positions and, more generally, about the qualities of the candidates. This is because of the inevitable difficulty of making precise political judgments and understanding associated returns, or because the parties deliberately obfuscate information to attract voters with different preferences and expand electoral support. More plausibly, in the wake of election campaigns, voters are conscious of their own and the parties' attitudes towards some popular issues, but might be uncertain about how they themselves and the parties stand towards more technical or less debated topics, and about the traits of the candidates other than those publicly advertised. Further, they may attempt to fill such gaps in information with various degrees of success and in different ways, depending on a priori inclination for certain parties, political sentiments, interest in specific issues, civic sense, attentional limits, participation in townhall debates, candidates' transparency, opinion makers, and media exposure. In turn, some individuals might become much more informed, others less, giving rise to heterogeneity in the public understanding of politics.

Despite the acknowledgement of the central role played by the sophistication of voters in determining voting patterns, only a few empirical works have attempted to take it into account while estimating a spatial voting framework. This has been done, for instance, through an additive, exogenous, and parametrically distributed error in the payoffs representing the evaluation mistakes made by voters, a parametric specification of the variance of the perceived party position across voters, or a parametric specification of the probability of being informed versus uninformed when voting (Degan and Merlo, 2011, and other references in Section 1). By contrast, our methodology permits us to incorporate uncertainty under weak assumptions on the latent, heterogeneous, and potentially endogenous process followed by voters to gather and evaluate information.

In particular, we assume that, when assessing the returns to voting for party $y \in \mathcal{Y}$, DM i is:

- Aware of the distances between their position and party y 's position on highly debated topics (for instance, EU integration). These distances are captured by the vector $Z_{iy} := |Z_i - Z_y|$.
- Potentially uncertain about their tastes towards party y 's opinion on more complicated and less media-covered issues (for instance, public expenditure management and reactions to pandemics) and towards the candidates' qualities that have been less advertised (for instance, disclosure of assets, liabilities, and any conflict of interests). These tastes are captured, in some aggregate way, by the vector $V_i := (V_{iy} : y \in \mathcal{Y})$. DM i has a prior on V_i . Further, DM i has access to a learning technology that allows them to become more informed about V_i .

We follow the literature on voting under uncertainty which typically models priors as normal distributions (Knight and Schiff, 2010; Matějka and Tabellini, 2021; Yuksel, 2022). In particular, we assume that V_i is distributed as a standard normal, independent of X_i . See also Feddersen and Pesendorfer (1997) and McMurray (2013) on the use of the common prior assumption and Bayesian rationality in models of voting. Lastly, we consider abstention as the base category and normalise its payoff to zero as, for example, in Knight and Schiff (2010).

5.2 Data

We estimate our model by using data on the UK general election held on 8 June 2017. Specifically, we use data from the British Election Study, 2017: Face-to-Face Post-Election Survey (Fieldhouse,

et al., 2018). The survey took place immediately after the election. It asks questions concerning key contemporary problems about political representation, accountability, and engagement, and aims to explain changes in party support. The interviewees constitutes an address-based random probability sample of eligible voters living in 468 wards in 234 Parliamentary Constituencies across England, Scotland, and Wales.

We believe that such data fit the framework described in Section 5.1 for four main reasons. First, the UK parties were clearly focused on the topic of Brexit, along with issues of public health and austerity, thus inducing potential uncertainty among voters with respect to many other factors (Hutton, 2017; Snowden and Demianyk, 2017). Second, the UK political scene is dominated by historical parties. Hence, past election outcomes and consequent behaviour of parties can justify the common prior assumption on V_i . Third, the survey reports the positions of the respondents on topics that were debated at length before the election, which we discuss more precisely below. The survey also asks respondents to state the parties’s positions with respect to those topics and the answers provided are substantially aligned. This suggests that there was no uncertainty among voters on those topics. Hence, they can be used to construct the vector Z_{iy} for each party $y \in \mathcal{Y}$, whose realization is assumed to be in the information set of voters. The survey does not contain data on other relevant factors that might have induced uncertainty among voters. Hence, it is natural to treat the realization of V_i as unobserved by the researcher. Fourth, the survey asks respondents to declare if they voted tactically. Only 2.16% of the respondents answer affirmatively. We drop them from our final sample, in order for the assumption that voters vote ideologically to apply.

To limit the impact of Scottish and Welsh independentist fronts on our results, we focus on the respondents who reside in England. We consider the answers of respondents on which party they have voted for among the Conservative Party, Labour Party, Liberal Democrats, United Kingdom Independence Party (UKIP), Green Party, and none.¹³

¹³The original questionnaire includes among the possible answers also the Scottish National Party (i.e., the Scottish nationalist social-democratic party in Scotland), Plaid Cymru (i.e., the Welsh nationalist social-democratic party in Wales), other unspecified minor parties, and “Refused to declare”. None of the respondents who reside in England have voted for the Scottish National Party. Only one of the respondents who reside in England has voted for Plaid Cymru. 4 respondents who reside in England have voted for other unspecified minor parties. Lastly, 3.52% of the respondents who reside in England have refused to declare who they have voted for. We have dropped all these observations.

We collect in Z_{iy} the distances between DM i 's position and party y 's position on four dimensions: EU integration, taxation and social care, income inequality, and left-right political orientation. More precisely, we select the answers of the respondents to the following questions (summarised with respect to the original version, for brevity):

1. [EU integration]: On a scale from 0 to 10, do you think that Britain should do all it can to unite fully with the European Union (0), or do all it can to protect its independence from the European Union (10)? Also provide the positions of the parties on the same scale.
2. [Taxation and social care]: On a scale from 0 to 10, do you think that government should cut taxes a lot and spend much less on health and social services (0), or that government should raise taxes a lot and spend much more on health and social services (10)? Also provide the positions of the parties on the same scale.
3. [Income inequality]: On a scale from 0 to 10, do you think that government should make much greater efforts to make people's incomes more equal (0), or that government should be much less concerned about how equal people's incomes are (10)? Also provide the positions of the parties on the same scale.
4. [Left-right political orientation] Where would you place yourself on a scale from 0 to 10 where 0 denotes left political attitudes and 10 denotes right political attitudes? Also provide the positions of the parties on the same scale.

Following the literature (Alvarez and Nagler, 1995; 1998; 2000; Alvarez, Nagler, and Bowler, 2000), we set party y 's position on dimensions 1-4 equal to the median placement of the party on each dimension across the sample, although as noticed above there is substantial alignment among the respondents' answers.

We collect in W_i some demographic characteristics of respondents. In particular, we focus on gender, socio-economic class, and total income before tax. Recall that γ_y captures the impact of W_i on the vote shares. We allow this impact to be heterogeneous across the parties. To be parsimonious on the number of parameters to estimate, we further parameterise γ_y by requiring that $\gamma_y := \gamma Z_y^{\text{LR}}$ for every party y , where Z_y^{LR} is the position of party y with regards to left-right

orientation. In other words, we assume that the aforementioned heterogeneity is driven by the position of each party in the left-right political spectrum.

In our final sample, 36.48% of people have voted for the Labour Party, 36.65% for the Conservative Party, 6.41% for the Liberal Democrats, 1.73% for UKIP, 1.56% for the Green Party, and 17.17% did not vote.¹⁴ Table 4 presents some descriptive statistics. The second column refers to the positions of the respondents on dimensions 1-4 and reports the mean (rounded to the nearest integer), median, and standard deviation across the sample. The remaining columns reports Z_y for each party y . As expected, the Conservative Party and UKIP are more right-wing, less concerned with income inequality, more Eurosceptic, and stronger supporters of low taxes and a minimal welfare state, than the Labour Party and the Green Party. The Liberal Democrats are more centrist.

	Self (Mean, Median, St.Dev.)			Conservative	Labour	Lib. Dem.	UKIP	Green
EU	5	5	3.355	7	4	3	10	3
Social care	7	7	2.051	5	7	6	4	6
Inequality	4	4	2.743	6	3	4	5	3
Left-right	5	5	2.059	8	2	5	9	3

Table 4: Descriptive statistics on the ideological positions.

The sample is gender balanced, with 48.97% of males and 51.03% of females. We assign label 1 to females and 0 to males. In the original data, the socio-economic class is divided into seven categories, following the Standard Occupation Classification 2010: professional occupations; managerial and technical occupations; skilled occupations - non-manual; skilled occupations - manual; partly skilled occupations; unskilled occupations; armed forces. To lessen the computational burden, we reorganise these categories into three groups. The first group is assigned label 0 and collects professional occupations, managerial and technical occupations, skilled occupations -

¹⁴Before omitting observations with missing data, the percentages are: 34.74% for the Labour Party, 33.78% for the Conservative Party, 5.39% for the Liberal Democrats, 1.91% for UKIP, 1.63% for the Green Party, and 22.56% did not vote. The survey seems slightly skewed towards supporters of the Labour Party. The actual vote shares in England were 41.9% for the Labour Party, 45.6% for the Conservative Party, 7.8% for the Liberal Democrats, 2.1% for UKIP, 1.9% for the Green Party. The election results led to a hung parliament and the Conservative Party formed a minority government supported by an agreement with the Northern Ireland’s Democratic Unionist Party. The vote shares of each party (including minorities) can be obtained, for example, here <https://www.bbc.co.uk/news/election/2017/results/england>. See also https://en.wikipedia.org/wiki/Opinion_polling_for_the_2017_United_Kingdom_general_election#2017 for opinion polls organised by various organisations to gauge voting intentions.

non-manual, and armed forces (68.04% of the sample). The second group is assigned label 1 and collects skilled occupations - manual and partly skilled occupations (29.42% of the sample). The third group is assigned label 2 and collects unskilled occupations (2.54% of the sample). Similarly, in the original data, the total income before tax is bracketed into 14 categories. We reorganise these categories into four groups, which we construct by approximately following the UK income tax rates. The first group is for income between £0 and £15,599 (21.78% of the sample). The second group is for income between £15,600 and £49,999 (51.68% of the sample). The third group is for income between £50,000 and £99,999 (21.45% of the sample). The fourth group is for income above £100,000 (5.09% of the sample). To each of the four groups, we assign as value the logarithm of the median income across the respondents belonging to that group (9.4727, 10.4282, 11.1199, and 12.6115, respectively). We summarise these numbers in Table 5.

Gender	Socio-economic class	Income (in log)
Males (0): 48.97%	First group (0): 68.04%	First group (9.4727): 21.78%
Females (1): 51.03%	Second group (1): 29.42%	Second group (10.4282): 51.68%
	Third group (2): 2.54%	Third group (11.1199): 21.45%
		Fourth group (12.6115): 5.09%

Table 5: Descriptive statistics on the demographic characteristics.

5.3 Results

We estimate the identified set of the 1×7 vector $\theta_0 := (\beta_0, \gamma_0)$ by evaluating a grid of 100,000 parameter values. The grid is constructed by exploring the parameter space, $\Theta \subseteq \mathbb{R}^7$, via the simulated annealing algorithm.¹⁵

To decide the order of the Bernstein polynomials, we take K_d constant across $d = 1, \dots, D$ and consider $K_d = 3, 5, 7, 10$. $D = 5$ because there are five parties and the payoff from abstention

¹⁵In the case of a high-dimensional vector of parameters, it is common in the partial identification literature to construct the grid of parameter values by using the simulated annealing algorithm (Ciliberto and Tamer, 2009). In particular, we proceed in four steps. First, we estimate θ_0 by maximum likelihood under the assumption that all DMs process the complete information structure and obtain the estimate $\hat{\theta}^{\text{com}}$. Second, we construct an Halton set of 10^6 points around $\hat{\theta}^{\text{com}}$. We draw 100 points at random from this set. We stack these points, together with $\hat{\theta}^{\text{com}}$, in an 101×7 matrix, A . Third, we minimise the test statistic $\text{TS}_n(\theta)$ defined in Appendix E with respect to θ by running the simulated annealing algorithm from each row of A as starting point and experimenting at different temperatures. We save every parameter value encountered in the course of the algorithm. We stack all the saved parameter values in a matrix G . Fourth, we draw 100,000 rows at random from G . Such 100,000 rows constitute our final grid of candidate parameter values.

is normalised to zero. For each of these four values of K_d , we construct an estimate of Θ^* , $\hat{\Theta}^*$, by replacing the empirical choice probabilities in (6) with their sample analogues and solving (6) for every realization x of X_i and parameter value θ in the grid. Table 6 reports the projections of $\hat{\Theta}^*$. As seen in Section 4, the width of the bounds tends to increase weakly with K_d . The bounds become approximately stable from $K_d = 5$ onwards. Therefore, in our next computations, we set $K_d = 5$ for $d = 1, \dots, D$.

K_d $d = 1, \dots, D$	β_1 EU	β_2 Social care	β_3 Inequality	β_4 Left-right	γ_1 Gender	γ_2 Class	γ_3 Income
3	[-0.3512, 0.0055]	[-0.0823, 0.0172]	[-0.1078, 0.0000]	[-0.3221, 0.0003]	[-0.3876, 0.3952]	[-0.2483, 0.1134]	[-0.1965, 0.4387]
5	[-0.3844, 0.0827]	[-0.1105, 0.0818]	[-0.1345, 0.0000]	[-0.3948, 0.0084]	[-0.4103, 0.4142]	[-0.2895, 0.1712]	[-0.2151, 0.4614]
7	[-0.3844, 0.0829]	[-0.1105, 0.0820]	[-0.1345, 0.0000]	[-0.3949, 0.0084]	[-0.4103, 0.4142]	[-0.2897, 0.1712]	[-0.2151, 0.4614]
10	[-0.3844, 0.0829]	[-0.1105, 0.0823]	[-0.1345, 0.0000]	[-0.3949, 0.0084]	[-0.4105, 0.4142]	[-0.2897, 0.1712]	[-0.2151, 0.4614]

Table 6: Projections of $\hat{\Theta}^*$ for different orders of the Bernstein polynomials.

Table 7 provides some computational details, as in Table 1 of Section 4. In particular, the fourth column shows the average CPU time to assess if (6) has a solution for a given (x, θ) , using the MOSEK solver for Matlab. The CPU time includes the calculation of the integrals $\{\gamma_{1,k,K}^{y,y',x}(\theta)\}$ by Monte Carlo integration, taking 10^2 random draws from the standard Normal distribution. The last column of Table 7 reports the rough total CPU time to construct $\hat{\Theta}^*$ based on exploiting 800 parallel workers from a computing cluster.

$K_d, d = 1, \dots, D$	Order polynomial $K := (K_d + 1)^D$	Unknowns in (6) $(D + 1)K$	CPU time per (x, θ)	Total CPU time
3	1,024	6,144	0.036 s	2 h
5	7,776	46,656	0.281 s	12 h
7	32,768	196,608	1.191 s	2 d
10	161,051	966,306	6.025 s	10 d

Note: $D = 5$. The CPU time per (x, θ) includes the calculation of $\{\gamma_{1,k,K}^{y,y',x}(\theta)\}$ using 10^2 random draws from the standard Normal distribution. The total CPU time is based on 800 parallel workers, a grid of 100,000 candidate values of θ , and 1,174 possible realizations of X_i .

Table 7: CPU time across different orders of the Bernstein polynomial.

Table 8 presents the inference results. In particular, the second and third columns report the maximum likelihood estimate of θ_0 ($\hat{\theta}^{\text{com}}$) and 95% confidence intervals ($C_{0.95}^{\text{com}}$), respectively, under the assumption that all DMs process the complete information structure. The third and fourth columns report the projections of $\hat{\Theta}^*$ and of the 95% confidence region for any $\theta \in \Theta^*$ ($C_{0.95}$),

		$\hat{\theta}^{\text{com}}$	$C_{0.95}^{\text{com}}$	$\hat{\Theta}^*$	$C_{0.95}$
β_1	EU	-0.0770 (0.0117)	[-0.0998, -0.0541]	[-0.3844, 0.0827]	[-0.5858, 0.0827]
β_2	Social care	-0.0064 (0.0173)	[-0.0402, 0.0274]	[-0.1105, 0.0818]	[-2.1726, 0.6404]
β_3	Inequality	-0.0342 (0.0138)	[-0.0613, -0.0070]	[-0.1345, 0.0000]	[-0.6414, 0.0000]
β_4	Left-right	-0.1507 (0.0140)	[-0.1782, -0.1232]	[-0.3948, 0.0084]	[-3.7394, 0.0086]
γ_1	Gender	-0.0071 (0.0091)	[-0.0248, 0.0107]	[-0.4103, 0.4142]	[-0.4142, 0.9715]
γ_2	Class	-0.0461 (0.0084)	[-0.0626, -0.0296]	[-0.2895, 0.1712]	[-0.2901, 0.1869]
γ_3	Income	0.0044 (0.0008)	[0.0028, 0.0060]	[-0.2151, 0.4614]	[-0.2172, 0.4618]

Table 8: Inference results.

respectively. $C_{0.95}$ is constructed following Andrews and Shi (2013), as outlined in Appendix E, based on 50 bootstrap samples.¹⁶

Under the assumption that voters are fully informed, all the β coefficients, except β_2 , are statistically different from zero at 5%. This suggests that DMs vote ideologically on the EU, inequality, and left-right dimensions. That is, the smaller the distance between DM i and party y 's ideological positions on those dimensions, the more likely DM i votes for party y , ceteris paribus. Further, β_4 has the highest absolute value magnitude among the β coefficients. That is, voters particularly disvalue casting their votes in favour of a party ideologically distant on the left-right axis. A one-unit increase in the ideological distance on the left-right axis produces a payoff decrease that is roughly 2, 23, and 4 times bigger than the payoff decrease produced by a one-unit increase in the ideological distance on the EU, social care, and inequality dimensions, respectively.

When we remain agnostic about voter sophistication, all the projections of $\hat{\Theta}^*$ for the β coefficients include zero. Therefore, differently from above, we cannot reject the possibility that the election outcomes have been generated under some combinations of information structures under which the ideological distances on the EU, social care, inequality, and left-right dimensions are irrelevant for voter preferences. Nevertheless, the model maintains enough identification power

¹⁶It took us around 12 days to obtain $C_{0.95}$ with $K_d = 5$ for $d = 1, \dots, D$.

to completely exclude positive values of β_3 and almost entirely positive values of β_4 . This is roughly confirmed by the projections of $C_{0.95}$. In particular, β_4 can have the highest absolute value magnitude among the β coefficients, in agreement with the maximum likelihood results. The fact that this finding on β_4 is robust to the restrictions on the information environment reflects several post-election descriptive studies run by political experts, which emphasise that the traditional left-right values, rather than specific policy issues, have been the main driver of the British electoral behaviour in 2017 (Hobolt, 2018).

The upper bounds of the projections for β_1 and β_2 are non-negligibly positive. While this aligns with the non-significance of β_2 under the complete information assumption ($C_{0.95}^{\text{com}}$ includes positive values), it is in contrast with the significance of β_1 under the complete information assumption and the 2017 election being often referred to as the “Brexit election” (Mellon, et al., 2018). Upon closer inspection, however, the inconclusive results on β_1 reflect that the 2017 pre-election period saw a substantial increase in the relationship between EU referendum choice and Labour versus Conservative vote choice, with a sort of alignment of the remain-leave axis with the traditional left-right axis. The parties with the clearest positions against Brexit (the Liberal Democrats) and in favour of Brexit (UKIP) lost many supporters. These switched *en masse* to the Labour Party, offering a “soft Brexit” and the Conservative Party, offering a “hard Brexit”, respectively (Mellon, et al., 2018; Heath and Goodwin, 2017). Such a tendency may have dampened the role of the distance in the Brexit sentiment in determining the preferences of voters, an insight that is not picked up under the complete information assumption.¹⁷

5.4 Counterfactuals

Information provision. The uncertainty about the payoffs resulting from voting can occur due to deliberate strategies of the candidates who “*becloud*” their characteristics and opinions “*in a fog of ambiguity*” (Downs, 1957, p.136), in order to expand the electoral support by attracting groups of voters with different political preferences (Campbell, 1983; Dahlberg, 2009; Tomz and van Houweling, 2009; Somer-Topcu, 2015). It remains unclear, however, to what extent such uncertainty affects the vote shares and, in turn, the election results. A better understanding is

¹⁷The projections of $\hat{\Theta}^*$ contain $\hat{\theta}^{\text{com}}$. This is in line with our identification results, according to which the sets of parameter values recovered under specific information structures belong to Θ^* when they are non-empty.

important for designing transparency policies that can improve citizens' welfare and parties' well-being. We investigate this question by imagining an omniscient mediator who implements a policy that gives voters complete information. This can be achieved, for instance, by organising school campaigns that develops political literacy; forcing candidates to publicly disclose their assets, liabilities, and criminal records; and enforcing a strict regulation regarding campaign spending and airtime.¹⁸ We simulate the counterfactual vote shares under complete information and study how they change compared to the factual scenario.

This question has been largely debated in the literature. As explained by Bartels (1996), political scientists have often answered it by arguing that a large population composed of possibly uninformed citizens acts as if it was fully informed, either because each voter uses cues and information shortcuts helping them to figure out what she needs to know about the political world; or because individual deviations from fully informed voting cancel out in a large election, producing the same aggregate election outcome as if voters were fully informed. Carpini and Keeter (1996) and Bartels (1996) are the first studies to use quantitative evidence to disconfirm such claims. They simulate counterfactual vote shares under complete information using data on the level of information of the survey respondents as rated by the interviewers or assessed by test items. Degan and Merlo (2011) propose an alternative approach, which is closer to ours. They consider a spatial model of voting with latent uncertainty. Differently from us, they estimate such a model by parametrically and exogenously specifying the probability that a voter is informed. They use their estimates to obtain counterfactual vote shares under complete information and find that making citizens more informed about electoral candidates decreases abstention. We contribute to this thread of the literature by providing a way to construct counterfactual vote shares under complete information, which neither requires the difficult task of measuring voters' level of information in the factual scenario, nor imposes parametric assumptions on the probability that a voter is informed.

Recall from Proposition 3 that, given a realization x of X_i and a parameter value θ , $\Psi^\ell(y|x;\theta)$ and $\Psi^h(y|x;\theta)$ are the minimum and maximum counterfactual probabilities of choosing alternative $y \in \mathcal{Y}$ when the DMs face the augmented decision problem $\{G(\theta, x), S^\dagger\}$, where S^\dagger is

¹⁸See, for example, Niemi and Junn (1998), Hooghe and Wilkenfeld (2007), and Pontes, Henn, and Griffiths (2019) on the impact of civic education on political engagement.

some unknown expansion of the information structure S implemented by the policy program, possibly heterogeneous across DMs. In the case analysed, S is the complete information structure and, hence, $S^\dagger = S$ for each DM. Moreover, observe that, when all DMs process the complete information structure, there is a unique model-implied choice distribution. Therefore, $\Psi^\ell(y|x; \theta) = \Psi^u(y|x; \theta) := \Psi(y|x; \theta)$. In light of all this, we compute

$$\widehat{\Delta}_y^* := \cup_{\theta \in \widehat{\Theta}^*} \sum_x (\Psi(y|x; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x) \quad \text{and} \quad \widehat{\Delta}_{y,0.95}^* := \cup_{\theta \in C_{0.95}} \sum_x (\Psi(y|x; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x),$$

for each $y \in \mathcal{Y}$, where $\widehat{\mathbb{P}}_Y(y|x)$ and $\widehat{\mathbb{P}}_X(x)$ are the sample probability of choosing y conditional on x and the sample probability of x , respectively. $\widehat{\Delta}_y^*$ and $\widehat{\Delta}_{y,0.95}^*$ are the estimated gains/losses in vote shares under complete information compared to the factual scenario.

Table 9 reveals that, when voters are fully informed, abstention drops with respect to the factual scenario. This shows that voters are more confident in choosing a party and aligns with the empirical results in Degan and Merlo (2011). We also find that the “losers” from the policy intervention are the two biggest parties, i.e., the Conservative Party and the Labour Party. Conversely, the “winners” from the policy intervention are the other minor parties, i.e., the Liberal Democrats and the Green Party. This suggests that there exists some payoff-relevant information unobserved by voters, and the historically dominant parties in the British political scene benefit the most from such uncertainty.¹⁹

	$\widehat{\Delta}_y^*$	$\widehat{\Delta}_{y,0.95}^*$
Abstention	[-0.1591, -0.0350]	[-0.1654, -0.0020]
Conservative	[-0.1722, -0.1293]	[-0.1722, -0.0190]
Labour	[-0.2316, -0.1716]	[-0.3317, -0.1211]
Lib. Dem	[0.1234, 0.1673]	[0.0857, 0.2507]
UKIP	[-0.0173, 0.2358]	[-0.1389, 0.3559]
Green	[0.1458, 0.1781]	[0.0469, 0.2056]

Table 9: Gains/losses in vote shares under complete information compared to the factual scenario.

Lastly, we quantify the voters’ maximum welfare cost of limited information, based on (10).

¹⁹The observed drop in abstention is not a mechanical effect of the normalisation to zero of the payoff from not voting. This is because the observed realization of V_{iy} could add to or subtract from the component of the payoff “observed pre-signal”, $\beta'Z_{iy} + \gamma'_y W_i$. Further, note that it could be that many/all voters in the population already observe the realization of V_i , in which case we should expect no significant change in the abstention share, regardless of the normalisation adopted.

In particular, we compute

$$\widehat{W}^* := \cup_{\theta \in \widehat{\Theta}^*} \Delta \mathbb{E}_\theta \quad \text{and} \quad \widehat{W}_{0.95}^* := \cup_{\theta \in C_{0.95}} \Delta \mathbb{E}_\theta$$

where

$$\Delta \mathbb{E}_\theta := \sum_x \left[\int_{\mathcal{Y}} u \left(\operatorname{argmax}_{y \in \mathcal{Y}} u(y, x, v; \theta), v \right) P_V(v) dv \right. \\ \left. - \int_{\mathcal{Y}} u \left(\operatorname{argmax}_{y \in \mathcal{Y}} \int_{\mathcal{Y}} u(y, x, v; \theta) P_V(v) dv, v \right) P_V(v) dv \right] \widehat{\mathbb{P}}_X(x),$$

Table 10 highlights that up to 1.1697 utility points would be gained, on average, if all voters were perfectly informed. To interpret the magnitude of this number, take β_4 equal to the lower bound of the corresponding projection of $\widehat{\Theta}^*$, i.e., $\beta_4 = -0.3948$ from Table 8. Observe that a utility increase of 1.1697 is equal to the utility increase generated by reducing the ideological distance from a given party on the left-right dimension of around three points.

\widehat{W}^*	$[0, 1.1697]$
$\widehat{W}_{0.95}^*$	$[0, 1.2387]$

Table 10: Maximum welfare cost of limited information.

Changes in covariates. Various political experts sustain that, while at the beginning of the 2017 election campaign the Conservative Party had a sizeable lead in the opinion polls over the Labour Party, as the campaign progressed the Labour Party recovered ground because it strengthened its left ideological position on social spending and nationalization of key public services (for example, Heath and Goodwin, 2017; Mellon, et al., 2018). To evaluate this, we reset the Labour Party’s placement on dimension 2 (social care) to be two points less (i.e., 5 instead of 7) and study if the Labour Party’s well-being worsens by looking at the change in its vote shares.

Recall from Proposition 4 that, given a realization x of X_i and a parameter value θ , $\Phi^\ell(y|x, x^\dagger; \theta)$ and $\Phi^u(y|x, x^\dagger; \theta)$ are the minimum and maximum counterfactual probabilities of choosing alternative $y \in \mathcal{Y}$ when the DMs face the augmented decision problem $\{G(\theta, x^\dagger), S\}$, where S is the unknown information structure S processed in the factual scenario, possibly heterogeneous across DMs, and x and x^\dagger are the covariate realizations before and after the intervention, respectively.

Given y denoting the Labour Party, we compute

$$\widehat{\Lambda}_y := \cup_{\theta \in \widehat{\Theta}^*} \left[\sum_{(x, x^\dagger)} (\Phi^\ell(y|x, x^\dagger; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x), \sum_{(x, x^\dagger)} (\Phi^u(y|x, x^\dagger; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x) \right],$$

and

$$\widehat{\Lambda}_{y,0.95} := \cup_{\theta \in C_{0.95}} \left[\sum_{(x, x^\dagger)} (\Phi^\ell(y|x, x^\dagger; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x), \sum_{(x, x^\dagger)} (\Phi^u(y|x, x^\dagger; \theta) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x) \right].$$

$\widehat{\Lambda}_y^*$ and $\widehat{\Lambda}_{y,0.95}^*$ are the estimated gains/losses in the Labour Party's vote share compared to the factual scenario. We also calculate the difference between the counterfactual and factual choice probabilities under the assumption that all voters process the complete information structure:

$$\widehat{\Lambda}_y^{\text{com}} := \sum_x (P_Y(y|x^\dagger; \widehat{\theta}^{\text{com}}) - \widehat{\mathbb{P}}_Y(y|x)) \widehat{\mathbb{P}}_X(x),$$

where $P_Y(y|x^\dagger; \widehat{\theta}^{\text{com}})$ is the counterfactual probability of choosing y conditional on x^\dagger under complete information and $\theta = \widehat{\theta}^{\text{com}}$, obtained using the standard Multivariate Probit formulas.

$\widehat{\Lambda}_y^{\text{com}}$ in Table 11 reveals that, when voters are assumed to be fully informed, weakening the social care position to 5 leads to a decrease in the Labour Party's vote share. Our results partly confirm this finding, as $\widehat{\Lambda}_y$ and $\widehat{\Lambda}_{y,0.95}$ mostly lie on the negative real line. This supports the claim that, by strengthening its left ideological position on the social care dimension, the Labour Party gained some votes during the election campaign.

$\widehat{\Lambda}_y^{\text{com}}$	-0.2283
$\widehat{\Lambda}_y$	[-0.3648, 0.0158]
$\widehat{\Lambda}_{y,0.95}$	[-0.5682, 0.0163]

Table 11: Gains/losses in the Labour Party's vote share compared to the factual scenario.

6 Conclusions

In this paper, we study identification of preferences in static single-agent discrete choice models where decision makers may be imperfectly informed about the state of the world. We leverage

the notion of 1BCE by BM16 to provide a tractable characterization of the sharp identified set. We make three main methodological contributions. First, by reinterpreting our framework as a 1-player game against nature, we provide insightful comparisons between the identification power of our framework and several many-player games. Second, we develop a formal procedure to practically construct the sharp identified set for the payoff parameters when the state of the world is continuous. Third, we characterize sharp bounds on the counterfactual choice probabilities when agents receive information about the state of the world via a policy program, which is an important question in the empirical literature on single-agent decision problems across different fields. The method developed in this paper is used to estimate a spatial voting model under weak assumptions on agents' information about the returns to voting for the 2017 UK general election. We show the usefulness of our methodology to quantify the consequences of imperfect information on the welfare of voters and parties.

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A Examples

Example 1. (*Additive random utility discrete choice models*) Consider the Nested Logit model with one nest collecting all goods but the outside option. The payoff function, u , is

$$u(y, X_i, V_i, \epsilon_i; \beta, \lambda) := \begin{cases} \beta^\top X_{iy} + \epsilon_i(\lambda) + \lambda V_{iy} & \text{if } y \in \mathcal{Y} \setminus \{0\}, \\ \eta_{i0} & \text{if } y = 0, \end{cases} \quad (\text{A.1})$$

where i indexes a generic DM, $\mathcal{Y} := \{0, 1, \dots, D\}$, 0 denotes the outside option, X_{iy} are DM-alternative specific covariates, and $(\epsilon_i(\lambda), V_{iy})$ are DM i 's tastes i.i.d. across y and independent of $X_i := (X_{i1}, \dots, X_{iD-1})$. The densities of $\epsilon_i(\lambda)$ and $V_i := (V_{i0}, \dots, V_{iD-1})$ are chosen to yield the familiar Nested Logit market share function, with $\lambda \in (0, 1]$. The researcher observes the choice made by DM i and the realization of X_i for a large sample of DMs.

Suppose that DM i observes the realization of $(X_i, \epsilon_i(\lambda))$ but might be uncertain about the realization of V_i . DM i has a prior on V_i conditional on $(X_i, \epsilon_i(\lambda))$, which is assumed to obey the Nested Logit parameterisation above. Further, DM i processes an information structure, S_i , to update their prior. Note that this framework reduces to the traditional Nested Logit model under the additional assumption that each agent in the population processes the complete information structure.

Similar considerations can be made for the (Multinomial) Logit model, Mixed Logit model, Probit model, etc. \diamond

Example 2. (*Risk aversion*) Consider a discrete choice model of insurance plans. Specifically, DM i faces an underlying risk of a loss (for example, a car accident) and can choose among L insurance plans. The loss event is denoted by C_i , where it takes value 1 if the loss event occurs and 0 otherwise. Each insurance plan $y \in \mathcal{Y}$ is characterised by a deductible, D_y , and a premium, P_{iy} . Further, DM i is endowed with some wealth (Wealth_i). The payoff function, u , belongs to

the CARA family, i.e., for each $y \in \mathcal{Y}$,

$$u(y, P_i, D, \text{Wealth}_i, r_i, C_i) := \begin{cases} \frac{1 - \exp[-r_i \times (\text{Wealth}_i - P_{iy} - D_y)]}{r_i} & \text{if } C_i = 1, r_i \neq 0, \\ \frac{1 - \exp[-r_i \times (\text{Wealth}_i - P_{iy})]}{r_i} & \text{if } C_i = 0, r_i \neq 0, \\ \text{Wealth}_i - P_{iy} - D_y & \text{if } C_i = 1, r_i = 0, \\ \text{Wealth}_i - P_{iy} & \text{if } C_i = 0, r_i = 0, \end{cases} \quad (\text{A.2})$$

where $P_i := (P_{i1}, \dots, P_{iL})$, $D := (D_1, \dots, D_L)$, and r_i is the coefficient of absolute risk aversion. r_i is often assumed distributed according to some parametric distribution such as the Beta distribution. The researcher observes the choice made by DM i and the realization of $(P_i, D, \text{Wealth}_i)$. In some cases, the researcher also observes the realization of C_i from ex-post data on claims.

Before choosing an insurance plan, DM i is aware of the realization of $(P_i, D, \text{Wealth}_i, r_i)$. However, DM i does not observe the realization of C_i because it is realised after the insurance plan choice has been made. Hence, following our general notation, $V_i := C_i$ and $(P_i, D, \text{Wealth}_i, r_i)$ are the variables in DM i 's information set in addition to the signal. DM i has a prior on V_i conditional on $(P_i, D, \text{Wealth}_i, r_i)$, which can be assumed to belong to some parametric family. For instance, one can use a simple Probit model or a more sophisticated Poisson-Gamma model (for an example of the latter see Barseghyan, Molinari, O' Donoghue, and Teitelbaum, 2013; Barseghyan, Molinari, and Teitelbaum, 2016). Further, DM i processes an information structure, S_i , to update their prior. S_i incorporates any extra private information on the risky event at the disposal of DM i , other than their level of risk aversion, and can arbitrarily depend on DM i 's risk aversion.

Under the additional restriction that each agent processes the null information structure, note that this framework collapses to the standard risk aversion setting considered in the empirical literature, where individuals have no extra private information on the risky event. \diamond

Example 3. (*Rational inattention*) Consider the rational inattention framework by Caplin and Dean (2015) and Matějka and McKay (2015). In that setting, the decision problem has two stages. In the first stage, DM i optimally chooses an information structure to update their prior. Although DM i is free to choose any information structure, attention is a scarce resource and

there is a cost of processing information. As a result, more informative signals are more costly. Such attentional costs are parameterised in various ways, for example, using the Shannon entropy (Sims, 2003) and the posterior-separable function (Caplin, Dean, and Leahy, 2019a). Formally, in the first stage, the DM chooses an information structure $S^* \in \mathcal{S}$ such that

$$S^* \in \operatorname{argmax}_{S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S}} \int_{\mathcal{T}} \int_{\mathcal{V}} \left[\max_{y \in \mathcal{Y}} \mathbb{E}_{S,t} u(y, V) \right] P_{T|V}(t|v) P_V(v) dv dt - C(S),$$

where $\mathbb{E}_{S,t} u(y, V)$ is the expected payoff from choosing $y \in \mathcal{Y}$ under the posterior induced by information structure S and signal realization t , and $C(S)$ represents the parameterised attentional costs associated with the information structure S . Then, in the second stage, the DM observes a signal realization, t , randomly drawn according to S^* . Lastly, the DM chooses alternative $y \in \mathcal{Y}$ maximising $\mathbb{E}_{S^*,t} u(y, V)$.

Note that our model nests the rational inattention framework because it remains completely silent on how agents choose information structures. In other words, we allow agents to choose information structures following the rational inattention protocol or any other procedure.

For an empirical counterpart of the rational inattention model, see Csaba (2018) and Brown and Jeon (2020). A few theoretical papers on rational inattention and, more generally, stochastic choice also grapple with the identification problem. However, those papers typically require either to have data on choices for every possible realization of the state of the world (for example, Caplin and Martin, 2015), or to have data on choices for multiple menus (for example, Lu, 2016; Lin, 2022).²⁰ These data are rarely available outside of laboratory experiments, especially for “complex” products as targeted here. Instead, in our framework the state of the world can be fully observed, partly observed, or fully unobserved by the researcher. Moreover, we assume to have data on choices for a single menu. In other words, we rely on less rich data. On one hand, this reduces the sources of identifying power and makes the task of deriving identification arguments more challenging; on the other hand, it allows for a wider applicability of our methodology.

Lastly, Hébert and Woodford (2018) and Morris and Strack (2019) consider continuous-time

²⁰According to our notation, having data on choices for multiple menus would consist of observing the choices of agents for multiple sets \mathcal{Y} .

models of sequential evidence accumulation and show that the resulting choice probabilities are identical to those of a static rational inattention model with posterior-separable attentional cost functions. That is, there is an equivalence between the information that is ultimately acquired in some search models and the information acquired in a static model of rational inattention, under a particular parameterisation of the attentional costs. Therefore, our setting also nests such search frameworks. \diamond

B Proofs

Proof of Theorem 1. Fix $\theta \in \Theta$. When \mathcal{V} and \mathcal{T} are not finite sets, P_V and $\{P_{T|V}(\cdot|v) : v \in \mathcal{V}\}$ should be interpreted as densities. As a preliminary step, it is useful to equivalently rewrite Definition 1 using the distribution of Y conditional on V as unknown, $\mathcal{P}_{Y|V} := \{P_{Y|V}(\cdot|v) : v \in \mathcal{V}\}$. $\mathcal{P}_{Y|V}$ is a 1BCE of $G(\theta)$ if:

1. It is *consistent*:

$$\sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) P_V(v; \theta_V) = P_V(v; \theta_V) \quad \forall v \in \mathcal{V}.$$

2. It is *obedient*:

$$\int_{\mathcal{V}} P_{Y|V}(y|v) P_V(v; \theta_V) (u(y, v; \theta_u) - u(y', v; \theta_u)) dv \geq 0, \quad \forall y' \in \mathcal{Y} \setminus \{y\}, \forall y \in \mathcal{Y}.$$

Observe that the *Consistency* requirement reduces to $\sum_{y \in \mathcal{Y}} P_{Y|V}(y|v) = 1$ for each $v \in \mathcal{V}$, which is necessary for $P_{Y|V}(y|v)$ to be a well-defined distribution. Hence, in what follows, we just consider the obedience constraint.

Let $\mathcal{P}_{Y|V}$ be a 1BCE of $G(\theta)$. We now show that there exists an information structure $S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S}$ and an optimal strategy $\mathcal{P}_{Y|T}$ of $\{G(\theta), S\}$ such that $\mathcal{P}_{Y|V}$ arises from $\mathcal{P}_{Y|T}$. Define $S := \{\mathcal{T}, \mathcal{P}_{T|V}\}$ such that

$$\mathcal{T} = \mathcal{Y}, \tag{B.1}$$

$$P_{T|V}(y|v) = P_{Y|V}(y|v) \quad \forall (y, v) \in \mathcal{Y} \times \mathcal{V}. \tag{B.2}$$

Let $\mathcal{P}_{Y|T}$ be a strategy of $\{G(\theta), S\}$ such that

$$P_{Y|T}(y|t) = \begin{cases} 1 & \text{if } y = t \\ 0 & \text{if } y \neq t \end{cases} \quad \forall (y, t) \in \mathcal{Y}^2. \quad (\text{B.3})$$

Observe that

$$P_{Y|V}(y|v) = \sum_{t \in \mathcal{Y}} P_{Y|T}(y|t) P_{T|V}(t|v) = P_{Y|V}(y|v) \quad \forall (y, v) \in \mathcal{Y} \times \mathcal{V}, \quad (\text{B.4})$$

where, in the first equality, we combine the formula of conditional probabilities with (B.1) and the fact that Y is independent of V conditional on T . In the second equality, we use (B.3) and (B.2). From (B.4), we conclude that $\mathcal{P}_{Y|V}$ arises from $\mathcal{P}_{Y|T}$. We need to show that $\mathcal{P}_{Y|T}$ is an optimal strategy of $\{G(\theta), S\}$. This is the case if

$$\int_{\mathcal{V}} P_{V|T}(v|y) u(y, v; \theta_u) dv \geq \int_{\mathcal{V}} P_{V|T}(v|y') u(y', v; \theta_u) dv, \quad (\text{B.5})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$ and for each $y \in \mathcal{Y}$. Using the formula of the posterior probability, (B.5) is equivalent to

$$\int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(y|v) u(y, v; \theta_u) dv \geq \int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(y'|v) u(y', v; \theta_u) dv, \quad (\text{B.6})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$ and $y \in \mathcal{Y}$. Further, by (B.2), (B.6) is equivalent to

$$\int_{\mathcal{V}} P_V(v; \theta_V) P_{Y|V}(y|v) u(y, v; \theta_u) dv \geq \int_{\mathcal{V}} P_V(v; \theta_V) P_{Y|V}(y'|v) u(y', v; \theta_u) dv, \quad (\text{B.7})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$ and $y \in \mathcal{Y}$. Lastly, observe that (B.7) holds by the *Obedience* requirement. Therefore, $\mathcal{P}_{Y|T}$ is an optimal strategy of $\{G(\theta), S\}$.

Let $\mathcal{P}_{Y|T}$ be an optimal strategy of $\{G(\theta), S\}$ for some $S \in \mathcal{S}$. We now show that $\mathcal{P}_{Y|V}$ arising from $\mathcal{P}_{Y|T}$ is a 1BCE of $G(\theta)$. By the definition of optimal strategy and using the formula of the

posterior probability, it holds that

$$\int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(t|v) u(y, v; \theta_u) dv \geq \int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(t|v) u(y', v; \theta_u) dv, \quad (\text{B.8})$$

for each $t \in \mathcal{T}$, $y' \in \mathcal{Y} \setminus \{y\}$, and $y \in \mathcal{Y}$ such that $P_{Y|T}(y|t) > 0$. By pre-multiplying both sides of (B.8) by $P_{Y|T}(y|t)$ and integrating over \mathcal{T} , we obtain

$$\int_{\mathcal{T}} P_{Y|T}(y|t) \int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(t|v) u(y, v; \theta_u) dv dt \geq \int_{\mathcal{T}} P_{Y|T}(y|t) \int_{\mathcal{V}} P_V(v; \theta_V) P_{T|V}(t|v) u(y', v; \theta_u) dv dt, \quad (\text{B.9})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$. By rearranging integrals, (B.9) is equivalent to

$$\int_{\mathcal{V}} P_V(v; \theta_V) \int_{\mathcal{T}} P_{Y|T}(y|t) P_{T|V}(t|v) u(y, v; \theta_u) dt dv \geq \int_{\mathcal{V}} P_V(v; \theta_V) \int_{\mathcal{T}} P_{Y|T}(y|t) P_{T|V}(t|v) u(y', v; \theta_u) dt dv, \quad (\text{B.10})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$. Using the independence of Y from V conditional on T , (B.10) is equivalent to

$$\int_{\mathcal{V}} P_V(v; \theta_V) P_{Y|V}(y|v) u(y, v; \theta_u) dv \geq \int_{\mathcal{V}} P_V(v; \theta_V) P_{Y|V}(y|v) u(y', v; \theta_u) dv, \quad (\text{B.11})$$

for each $y' \in \mathcal{Y} \setminus \{y\}$. Lastly, observe that (B.11) is just the *Obedience* requirement. Therefore, $P_{Y|V}$ that arises from $\mathcal{P}_{Y|T}$ is a 1BCE of $G(\theta)$.

Proof of Proposition 1. Fix $\theta \in \Theta$. We show that if $P_Y \in \mathcal{Q}(\theta)$, then $P_Y \in \mathcal{R}(\theta)$. If $P_Y \in \mathcal{Q}(\theta)$, then, by definition of $\mathcal{Q}(\theta)$, there exists $P_{Y,V} \in \mathcal{W}(\theta)$ such that P_Y arises from $P_{Y,V}$. By Theorem 1, it follows that there exists $S \in \mathcal{S}$ and $\mathcal{P}_{Y|T} \in \mathcal{R}(\theta, S)$ such that $P_{Y,V}$ arises from $\mathcal{P}_{Y|T}$. Thus, P_Y arises from $\mathcal{P}_{Y|T}$. Therefore, by the definition of $\mathcal{R}(\theta)$, $P_Y \in \mathcal{R}(\theta)$.

Conversely, we show that $P_Y \in \mathcal{R}(\theta)$, then $P_Y \in \mathcal{Q}(\theta)$. First, let $\tilde{\mathcal{R}}(\theta) \subseteq \mathcal{R}(\theta)$ be the non-convexified set of model's predictions under θ while remaining agnostic about information structures. That is,

$$\tilde{\mathcal{R}}(\theta) := \left\{ P_Y \in \Delta(\mathcal{Y}) : P_Y(y) = \int_{\mathcal{V}} \int_{\mathcal{T}} P_{Y|T}(y|t) P_{T|V}(t|v) P_V(v; \theta_V) dt dv \quad \forall y \in \mathcal{Y}, \right. \\ \left. \mathcal{P}_{Y|T} \in \mathcal{R}(\theta, S), S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S} \right\},$$

Take $P_Y \in \tilde{\mathcal{R}}(\theta)$. Then, by the definition of $\tilde{\mathcal{R}}(\theta)$, there exists $S \in \mathcal{S}$ and $\mathcal{P}_{Y|T} \in \mathcal{R}(\theta, S)$ such that P_Y arises from $\mathcal{P}_{Y|T}$. By Theorem 1, it follows that there exists $P_{Y,V} \in \mathcal{W}(\theta)$ such that $\mathcal{P}_{Y|T}$ arises from $P_{Y,V}$. Thus, P_Y arises from $P_{Y,V}$. Hence, by the definition of $\mathcal{Q}(\theta)$, $P_Y \in \mathcal{Q}(\theta)$. Now, take any K elements from $\tilde{\mathcal{R}}(\theta)$. Denote such elements by $P_Y^1 \in \tilde{\mathcal{R}}(\theta), \dots, P_Y^K \in \tilde{\mathcal{R}}(\theta)$. Given the arguments above, it holds that $P_Y^1 \in \mathcal{Q}(\theta), \dots, P_Y^K \in \mathcal{Q}(\theta)$. Moreover, any convex combination of P_Y^1, \dots, P_Y^K belongs to $\mathcal{Q}(\theta)$ because $\mathcal{Q}(\theta)$ is convex. Therefore, every $P_Y \in \mathcal{R}(\theta)$ is also contained in $\mathcal{Q}(\theta)$.

We can conclude that $\mathcal{R}(\theta) = \mathcal{Q}(\theta)$ for each $\theta \in \Theta$. This implies $\Theta^* = \Theta^{**}$.

Proof of Proposition 2. Consider part (i). We want to show that, for every \mathcal{Y} and \mathcal{V} , there exists data, $\mathbb{P}_Y \in \Delta(\mathcal{Y})$, such that $\mathbb{P}_Y \notin \mathcal{R}(u, P_V)$ for some $(u, P_V) \in \mathcal{U} \times \Delta(\mathcal{V})$. For simplicity of notation, we consider the case where $\mathcal{Y} = \{0, 1\}$ and \mathcal{V} is finite. The extension to $|\mathcal{Y}| > 2$ and non-finite \mathcal{V} is straightforward and we discuss it towards the end. Fix $\mathbb{P}_Y \in \Delta(\mathcal{Y})$, $u \in \mathcal{U}$, and $P_V \in \Delta(\mathcal{V})$. By Proposition 1, the pair (u, P_V) is such that $\mathbb{P}_Y \in \mathcal{R}(u, P_V)$ if and only if the following linear program has a solution with respect to $P_{Y,V} : \{0, 1\} \times \mathcal{V} \rightarrow \mathbb{R}$:

$$\begin{aligned}
\text{Consistency:} \quad & \sum_{y \in \{0,1\}} P_{Y,V}(y, v) = P_V(v) \quad \forall v \in \mathcal{V}, \\
\text{Obedience:} \quad & \sum_{v \in \mathcal{V}} P_{Y,V}(1, v)(u(1, v) - u(0, v)) \geq 0, \quad \sum_{v \in \mathcal{V}} P_{Y,V}(0, v)(u(0, v) - u(1, v)) \geq 0, \\
\text{Probability requirements:} \quad & P_{Y,V}(1, v) \geq 0, P_{Y,V}(0, v) \geq 0 \quad \forall v \in \mathcal{V}, \quad \sum_{y \in \{0,1\}, v \in \mathcal{V}} P_{Y,V}(y, v) = 1, \\
\text{Data match:} \quad & \sum_{v \in \mathcal{V}} P_{Y,V}(y, v) = \mathbb{P}_Y(y), \quad \forall y \in \{0, 1\}.
\end{aligned}$$

We simplify our system's notation by defining:

$$x_{y,v} := P_{Y,V}(y, v), \quad \forall v \in \mathcal{V}, \forall y \in \{0, 1\},$$

$$x_y := (x_{y,v} : v \in \mathcal{V}), \quad \forall y \in \{0, 1\},$$

$$z_{y,v} := u(y, v), \quad \forall v \in \mathcal{V}, \forall y \in \{0, 1\},$$

$$z_y := (z_{y,v} : v \in \mathcal{V}), \quad \forall y \in \{0, 1\},$$

$$z := z_1 - z_0,$$

$$q_y := \mathbb{P}_Y(y), \quad \forall y \in \{0, 1\},$$

$$w_v := P_V(v), \quad \forall v \in \mathcal{V},$$

$$w := (w_v : v \in \mathcal{V}),$$

$$\mathbb{1}_{|\mathcal{V}|} \text{ is a column vector of ones of size } |\mathcal{V}| \times 1.$$

Hence, our system becomes

$$\sum_{y \in \{0,1\}} x_{y,v} = w_v \quad \forall v \in \mathcal{V}, \tag{B.12}$$

$$\sum_{v \in \mathcal{V}} x_{y,v} = q_y \quad \forall y \in \{0, 1\}, \tag{B.13}$$

$$\sum_{v \in \mathcal{V}} x_{1,v} (z_{1,v} - z_{0,v}) \geq 0, \tag{B.14}$$

$$\sum_{v \in \mathcal{V}} x_{0,v} (z_{1,v} - z_{0,v}) \leq 0, \tag{B.15}$$

$$\sum_{y \in \{0,1\}, v \in \mathcal{V}} x_{y,v} = 1, \tag{B.16}$$

$$0 \leq x_{y,v} \leq 1 \quad \forall y \in \{0, 1\}, \forall v \in \mathcal{V}. \tag{B.17}$$

Observe that the upper bound in (B.17) is implied by (B.16) and the lower bound in (B.17). Also, (B.16) is implied by (B.12) or (B.13). Hence, the problem reduces to

$$x_0 + x_1 = w, \tag{B.18}$$

$$x_0 \mathbb{1}_{|\mathcal{V}|} = q_0, \tag{B.19}$$

$$x_1 \mathbb{1}_{|\mathcal{V}|} = q_1, \tag{B.20}$$

$$x_1 z^\top \geq 0, \tag{B.21}$$

$$x_0 z^\top \leq 0, \tag{B.22}$$

$$x_0 \geq 0, \tag{B.23}$$

$$x_1 \geq 0. \tag{B.24}$$

From (B.18), $x_1 = w - x_0$. We plug this into (B.20), (B.21), and (B.24). The system becomes

$$x_0 \mathbb{1}_{|\mathcal{V}|} = q_0, \tag{B.25}$$

$$w \mathbb{1}_{|\mathcal{V}|} - q_1 = x_0 \mathbb{1}_{|\mathcal{V}|} \tag{B.26}$$

$$wz^\top \geq x_0 z^\top, \tag{B.27}$$

$$x_0 z^\top \leq 0, \tag{B.28}$$

$$x_0 \geq 0, \tag{B.29}$$

$$x_0 \leq w. \tag{B.30}$$

Observe that $w \mathbb{1}_{|\mathcal{V}|} = 1$. Therefore, (B.26) is simply $x_0 \mathbb{1}_{|\mathcal{V}|} = 1 - q_1$. Hence, (B.25) is equal to (B.26) and the system reduces further to

$$x_0 \mathbb{1}_{|\mathcal{V}|} = q_0, \tag{B.31}$$

$$x_0 z^\top \leq \min\{0, w z^\top\}, \tag{B.32}$$

$$0 \leq x_0 \leq w. \tag{B.33}$$

Note that this system has no solutions when $z > 0$ and $q_0 > 0$. In fact, suppose $|\mathcal{V}| := \{v_a, v_b\}$. By writing (B.32) explicitly, we get

$$x_{0,v_a}(z_{1,v_a} - z_{0,v_a}) + x_{0,v_b}(z_{1,v_b} - z_{0,v_b}) \leq \min\{0, w_{v_a}(z_{1,v_a} - z_{0,v_a}) + w_{v_b}(z_{1,v_b} - z_{0,v_b})\}.$$

Observe that if $z > 0$, then $w_{v_a}(z_{1,v_a} - z_{0,v_a}) + w_{v_b}(z_{1,v_b} - z_{0,v_b}) > 0$. Thus, $\min\{0, w_{v_a}(z_{1,v_a} - z_{0,v_a}) + w_{v_b}(z_{1,v_b} - z_{0,v_b})\} = 0$. Since $z > 0$, $x_{0,v_a}(z_{1,v_a} - z_{0,v_a}) + x_{0,v_b}(z_{1,v_b} - z_{0,v_b}) = 0$ if and only if $x_{0,v_a} = x_{0,v_b} = 0$, which violates (B.31) when $q_0 > 0$. Therefore, we conclude that, when $|\mathcal{Y}| = 2$ and \mathcal{V} finite, there exists data, $\mathbb{P}_Y \in \Delta(\mathcal{Y})$, such that $\mathbb{P}_Y \notin \mathcal{R}(u, P_V)$ for some $(u, P_V) \in \mathcal{U} \times \Delta(\mathcal{V})$.

We can apply the same intuition when $|\mathcal{Y}| > 2$ and \mathcal{V} is not finite, after having replaced vectors with functions and considering the inner product between two vectors a, b as $\int_{\mathcal{Y}} a(v)b(v)dv$. Therefore, we can generically conclude that, for every \mathcal{Y} and \mathcal{V} , there exists data, $\mathbb{P}_Y \in \Delta(\mathcal{Y})$, such that $\mathbb{P}_Y \notin \mathcal{R}(u, P_V)$ for some $(u, P_V) \in \mathcal{U} \times \Delta(\mathcal{V})$ and, so, Θ^* is a strict subset of the parameter space, $\mathcal{U} \times \Delta(\mathcal{V})$.

Consider part (ii). Assume that all DMs process the complete information structure. Under such assumption, for each $(\tilde{u}, P_V) \in \tilde{\mathcal{U}} \times \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$, $\mathcal{R}(\tilde{u}, P_V) := \{P_{Y;\tilde{u},P_V}\}$, where $P_{Y;\tilde{u},P_V}$ is the *unique* model-implied choice distribution. Observe that:

1. Haile, Hortaçsu, and Kosenok (2008) investigate the identification power of the notion of quantile response equilibrium (QRE) in N -player games with $N \geq 1$ and additively separable shocks. In their Theorem 1, they show that QRE imposes no restrictions on behaviour. That is, for any systematic payoff and distribution of observed behaviour, there exists a density of the additively separable unobserved terms entering payoffs such that a QRE generates the distribution of observed behaviour. When $N = 1$, the QRE framework reduces to an additive random utility discrete choice model (Footnote 9, p.184, Haile, Hortaçsu, and Kosenok, 2008). Hence, we can apply Theorem 1 of Haile, Hortaçsu, and Kosenok (2008) to our model and show that, when all DMs process the complete information structure, any \tilde{u} rationalises the empirical choice distribution. Formally, for every $P_Y \in \Delta(\mathcal{Y})$ and $\tilde{u} \in \tilde{\mathcal{U}}$, there exists $P_V \in \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ such that $P_Y = P_{Y;\tilde{u},P_V}$.
2. Galichon and Salanié (2022) study identification in 2-sided matching models with additively

separable shocks. In their Proposition 1, they show that 2-sided matching models are under-identified. Specifically, for any density of the additively separable unobserved terms entering payoffs and distribution of observed behaviour, there exists a systematic payoff such that the model-implied stable match generates the distribution of observed behaviour. The result applies also to the 1-sided case of an additive random utility discrete choice model. Hence, we can apply Proposition 1 of Galichon and Salanié (2022) to our model and show that, when all DMs process the complete information structure, any P_V rationalises the empirical choice distribution. Formally, for every $P_Y \in \Delta(\mathcal{Y})$ and $P_V \in \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$, there exists $\tilde{u} \in \tilde{\mathcal{U}}$ such that $P_Y = P_{Y;\tilde{u},P_V}$.

Without restrictions on information structures, $\mathcal{R}(\tilde{u}, P_V)$ is not a singleton and $P_{Y;\tilde{u},P_V} \in \mathcal{R}(\tilde{u}, P_V)$. Therefore, by 1, for every $P_Y \in \Delta(\mathcal{Y})$ and $\tilde{u} \in \tilde{\mathcal{U}}$, there exists $P_V \in \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ such that $P_Y \in \mathcal{R}(\tilde{u}, P_V)$. This implies that the projection of Θ^* on $\tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ is equal to $\tilde{\mathcal{U}}$ and part (ii)-(a) holds. Similarly, by 2, for every $P_Y \in \Delta(\mathcal{Y})$ and $P_V \in \tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$, there exists $\tilde{u} \in \tilde{\mathcal{U}}$ such that $P_Y \in \mathcal{R}(\tilde{u}, P_V)$. This implies that the projection of Θ^* on $\tilde{\mathcal{U}}$ is equal to $\tilde{\Delta}(\mathbb{R}^{|\mathcal{Y}|})$ and part (ii)-(b) holds.

Proof of Proposition 3. Fix $\theta \in \Theta$. We redefine the notions of baseline decision problem and augmented decision problem. Let $\{G(\theta), S\}$ be the baseline decision problem that the DM faces when processing the baseline information structure $S := \{\mathcal{T}, \mathcal{P}_{T|V}\} \in \mathcal{S}$ implemented by the policy program. Let $\{G(\theta), S^\dagger\}$ be the augmented decision problem that the DM faces when processing the expanded information structure $S^\dagger := \{\mathcal{T}^\dagger, \mathcal{P}_{T^\dagger|V}\} \in \mathcal{S}$ obtained as a combination of S and some $S^\diamond := \{\mathcal{T}^\diamond, \mathcal{P}_{T^\diamond|V}\} \in \mathcal{S}$. Let Y^\dagger denote the DM's choice in the counterfactual scenario. An optimal strategy in $\{G(\theta), S^\dagger\}$ is a distribution of Y^\dagger conditional on (T, T^\diamond) , $\mathcal{P}_{Y^\dagger|T, T^\diamond} := \{P_{Y^\dagger|T, T^\diamond}(\cdot|t, t^\diamond) : (t, t^\diamond) \in \mathcal{T} \times \mathcal{T}^\diamond\}$, such that, for each $(t, t^\diamond) \in \mathcal{T} \times \mathcal{T}^\diamond$, the DM maximises their expected utility by choosing any alternative $y^\dagger \in \mathcal{Y}$ featuring $P_{Y^\dagger|T, T^\diamond}(y^\dagger|t, t^\diamond) > 0$.

Next, we define a 1BCE of $\{G(\theta), S\}$. $P_{Y^\dagger, V, T} \in \Delta(\mathcal{Y} \times \mathcal{V} \times \mathcal{T})$ is a 1BCE of $\{G(\theta), S\}$ if it is consistent, i.e., $\sum_{y^\dagger \in \mathcal{Y}} P_{Y^\dagger, V, T}(y^\dagger, v, t) = P_V(v; \theta_V) P_{T|V}(t|v) \forall (v, t) \in \mathcal{V} \times \mathcal{T}$, and obedient, i.e., $\int_{\mathcal{V}} P_{Y^\dagger, V, T}(y^\dagger, v, t)(u(y^\dagger, v; \theta_u) - u(k^\dagger, v; \theta_u)) dv \geq 0 \forall y^\dagger \in \mathcal{Y}$, $k^\dagger \in \mathcal{Y} \setminus \{y^\dagger\}$, and $t \in \mathcal{T}$.

We now state Theorem 1 of BM16, which still applies to the new setting. This theorem shows

that $P_{Y^\dagger, V, T}$ is a 1BCE of $\{G(\theta), S\}$ if and only if, for some expansion S^\dagger of S , there exists an optimal strategy of $\{G(\theta), S^\dagger\}$ that induces $P_{Y^\dagger, V, T}$.

It follows immediately that $\mathcal{P}_{Y^\dagger}^*(h)$ can be characterised via the union across $\theta \in \Theta^*$ of the set of 1BCE $P_{Y^\dagger, V, T}$ of $\{G(\theta), S\}$. In particular, (7) is the *Consistency* constraint, (8) is the *Obedience* constraint, and (9) ensures that $P_{Y^\dagger, V, T}$ is a well-defined distribution.

Proof of Proposition 4. Fix $\theta \in \Theta$. From Theorem 1 in Bergemann, Brooks, and Morris (2022), $\mathcal{Q}_{Y^\dagger|x, x^\dagger}^*(h)$ can be characterised via the union across $\theta \in \Theta^*$ of the set of 1BCE $P_{Y, Y^\dagger, V|x, x^\dagger}$ of the linked baseline decision problem $\{G(\theta, x), G(\theta, x^\dagger)\}$. In particular, (11) is the *Consistency* constraint, (12) and (13) are the *Obedience* constraints for each decision problem, (14) ensures that $P_{Y, Y^\dagger, V|x, X^\dagger}$ is a well-defined distribution, and (15) imposes that the factual choice probabilities are equal to the empirical ones.

C Linear program with observed and unobserved heterogeneity

In this section, we readapt (6) to the case where there are discrete covariates X (observed by the researcher) and discrete heterogeneity ϵ (unobserved by the researcher) which enter the DM's information set together with the signal. For simplicity, we assume that X is independent of (V, ϵ) . However, note that the computational procedure goes through even when X is not independent of (V, ϵ) , and so the distribution of (V, ϵ) is X -specific. Let \mathcal{X} and \mathcal{E} be the supports of X and ϵ , respectively. Let $P_\epsilon(\cdot; \theta_\epsilon)$ be the distribution of ϵ , indexed by the structural parameter $\theta_\epsilon \in \Theta_\epsilon$. For each $e \in \mathcal{E}$, let $P_{V|\epsilon}(\cdot|e; \theta_V)$ be the distribution of V conditional on $\epsilon = e$. For a given $\theta := (\theta_U, \theta_V, \theta_\epsilon) \in \Theta := \Theta_U \times \Theta_V \times \Theta_\epsilon$, the researcher needs to verify if the following finite-

dimensional linear program has a solution with respect to $\lambda := (\lambda_{k,K}^{y,x,e} : (k, y, x, e) \in \mathcal{K} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{E})$:

$$\begin{aligned}
\text{Obedience:} \quad & \sum_{k \in \mathcal{K}} \lambda_{k,K}^{y,x,e} \gamma_{1,k,K}^{y,y',x,e}(\theta_V, \theta_U) \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \forall x \in \mathcal{X}, \forall e \in \mathcal{E}, \\
\text{Probability requirements:} \quad & \lambda_{k,K}^{y,x,e} \geq 0 \quad \forall k \in \mathcal{K}, \forall y \in \mathcal{Y}, \forall x \in \mathcal{X}, \forall e \in \mathcal{E}, \quad \sum_{y \in \mathcal{Y}} \lambda_{k,K}^{y,x,e} = 1 \quad \forall k \in \mathcal{K}, \forall x \in \mathcal{X}, \forall e \in \mathcal{E}, \\
\text{Data match:} \quad & \sum_{e \in \mathcal{E}} \sum_{k \in \mathcal{K}} \lambda_{k,K}^{y,x,e} \gamma_{2,k,K}^e(\theta_V; \theta_\epsilon) = \mathbb{P}_Y(y) \quad \forall y \in \mathcal{Y}, \forall x \in \mathcal{X},
\end{aligned} \tag{C.1}$$

where

$$\begin{aligned}
\gamma_{1,k,K}^{y,y',x,e}(\theta_V, \theta_U) &:= \int_{\mathcal{V}} a_{k,K}(v) P_{V|\epsilon}(v|e; \theta_V) (u(y, x, v, e; \theta_u) - u(y', x, v, e; \theta_u)) dv, \\
\gamma_{2,k,K}^e(\theta_V, \theta_\epsilon) &:= P_\epsilon(e; \theta_\epsilon) \int_{\mathcal{V}} a_{k,K}(v) P_{V|\epsilon}(v|e; \theta_V) dv.
\end{aligned}$$

Note that (C.1) is separable across $x \in \mathcal{X}$ and, therefore, can be solved separately across $x \in \mathcal{X}$.

D How consideration sets arise in our framework

Let $\mathcal{P}_{Y|T}$ be an optimal strategy of the augmented decision problem $\{G, S\}$ as defined in Section 2. Following Caplin, Dean, and Leahy (2019), the DM's consideration set, \mathcal{C} , arises endogenously from $\mathcal{P}_{Y|T}$. In particular, \mathcal{C} collects every alternative such that the subset of the signal's support inducing the DM to choose that alternative has positive measure:

$$\mathcal{C} \equiv \{y \in \mathcal{Y} : \int_{\mathcal{T}} P_{Y|T}(y|t) \int_{\mathcal{V}} P_{T|V}(t|v) P_V(v) dv dt > 0\},$$

Observe that consideration sets can be heterogeneous across agents as we leave the conditional signal densities unrestricted.

Therefore, this paper also relates to the econometric literature on discrete choice models with consideration sets. For some recent contributions see, for example, Cattaneo, Ma, Masatlioglu, and Suleymanov (2020); Dardanoni, Manzini, Mariotti, and Tyson (2020); Abaluck and Adams (2021); Barseghyan, Coughlin, Molinari, and Teitelbaum (2021); Barseghyan, Molinari, and Thirkettle (2021); Crawford, Griffith, and Iaria (2021); Dardanoni, Manzini, Mariotti, Petri, and Tyson (2022)). Yet, there is an important difference between the literature on consideration sets and

this paper. The consideration set literature focuses on (partially) identifying the consideration probabilities and payoffs under the assumption that the DM perfectly knows the payoff generated by each alternative in their consideration set. Instead, our setting partially identifies the payoffs (and does not identify the consideration probabilities) while allowing the DM not to be fully aware of the payoff generated by each alternative in their consideration set. Therefore, the two frameworks are non-nested and can answer different questions.

E Inference

We enrich the model of Section 2 by assuming that there are discrete exogenous covariates X with support \mathcal{X} which enter the DM's information set together with the signal, as it is the case in the empirical application. We assume that the researcher has a sample of i.i.d. observations, $\{Y_i, X_i\}_{i=1}^n$. Given $\alpha \in (0, 1)$, this section illustrates how to construct a uniformly asymptotically valid $(1-\alpha)\%$ confidence region, $C_{1-\alpha}$, for any $\theta \in \Theta^*$. In particular, we suggest to reformulate our problem using conditional moment inequalities and apply the generalised moment selection procedure by Andrews and Shi (2013) (hereafter, AS13), as detailed in Appendix B.1 of Beresteanu, Molchanov, and Molinari (2011) (hereafter, BMM11).

$C_{1-\alpha}$ is obtained by running a test with null hypothesis $H_0 : \theta_0 = \theta$, for every $\theta \in \Theta$, and then collecting all the values of θ which are not rejected. For a given θ , the test rejects H_0 if $\text{TS}_n(\theta) > \hat{c}_{n,1-\alpha}(\theta)$, where $\text{TS}_n(\theta)$ is a test statistic and $\hat{c}_{n,1-\alpha}(\theta)$ is a corresponding critical value. Thus,

$$C_{1-\alpha} \equiv \{\theta \in \Theta: \text{TS}_n(\theta) \leq \hat{c}_{n,1-\alpha}(\theta)\}. \quad (\text{E.1})$$

The remainder of the section explains how to compute $\text{TS}_n(\theta)$ and $\hat{c}_{n,1-\alpha}(\theta)$ for any given $\theta \in \Theta$.

In order to define the test statistic, $\text{TS}_n(\theta)$, let us first rewrite the linear programming (3) as a collection of conditional moment inequalities. To do so, we label the elements of \mathcal{Y} as $y^1, \dots, y^{|\mathcal{Y}|-1}, y^{|\mathcal{Y}|}$. We denote by $\mathbb{B}^{|\mathcal{Y}|-1}$ the unit ball in $\mathbb{R}^{|\mathcal{Y}|-1}$. For each $\theta \in \Theta$ and $x \in \mathcal{X}$, $\mathcal{Q}(\theta, x)$ is the set of model-implied choice distributions, $P_{Y|X}(\cdot|x)$, conditional on $X_i = x$. $\mathbb{P}_{Y|X}(\cdot|x)$ is the empirical choice distribution conditional on $X_i = x$.

Proposition E.1. (*Conditional moment inequalities*) For each $\theta \in \Theta$, $\theta \in \Theta^*$ if and only if

$$\mathbb{E}[m(Y_i, X_i; b, \theta) | X_i = x] \leq 0 \quad \forall b \in \mathbb{B}^{|\mathcal{Y}|-1}, \forall x \in \mathcal{X},$$

where

$$m(Y_i, x; b, \theta) \equiv b^\top \begin{pmatrix} \mathbb{1}\{Y_i = y^1\} \\ \dots \\ \mathbb{1}\{Y_i = y^{|\mathcal{Y}|-1}\} \end{pmatrix} - \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} b^\top \begin{pmatrix} P_{Y|X}(y^1|x) \\ \dots \\ P_{Y|X}(y^{|\mathcal{Y}|-1}|x) \end{pmatrix}.$$

◇

Proposition E.1 comes from the fact that, following BMM11, one can express the condition $\mathbb{P}_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)$ as

$$b^\top \mathbb{P}_{Y|X}(\cdot|x) - \sup_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} b^\top P_{Y|X}(\cdot|x) \leq 0 \quad \forall b \in \mathbb{R}^{|\mathcal{Y}|}, \quad (\text{E.2})$$

where the map

$$b \in \mathbb{R}^{|\mathcal{Y}|} \mapsto \sup_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} b^\top P_{Y|X}(\cdot|x) \in \mathbb{R},$$

is the support function of $\mathcal{Q}(\theta, x)$. By exploiting the positive homogeneity of the support function and some algebraic manipulations, (E.2) is equal to the collection of conditional moment inequalities listed in Proposition E.1.

Second, we rewrite the conditional moment inequalities in Proposition E.1 as unconditional moment inequalities. Here, we use Lemma 2 in AS13 which shows that conditional moment inequalities can be transformed into unconditional moment inequalities by choosing appropriate instruments, $h \in \mathcal{H}$, where \mathcal{H} is a collection of instruments and h is a function of X_i . Thus,

$$\theta \in \Theta^* \Leftrightarrow \mathbb{E}[m(Y_i, X_i; b, \theta, h)] \leq 0 \quad \forall b \in \mathbb{B}^{|\mathcal{Y}|-1}, \forall h \in \mathcal{H} \text{ a.s.}, \quad (\text{E.3})$$

where

$$m(Y_i, X_i; b, \theta, h) \equiv m(Y_i, X_i; b, \theta) \times h(X_i).$$

Third, observe that $\mathbb{E}[m(Y_i, X_i; b, \theta, h)]$ evaluated at $b \equiv 0_{|\mathcal{Y}|-1}$ is 0. Therefore, (E.3) is equivalent

to

$$\theta \in \Theta^* \Leftrightarrow \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} \mathbb{E}[m(Y_i, X_i; b, \theta, h)] = 0 \quad \forall h \in \mathcal{H} \text{ a.s.}$$

In light of the three steps above, following Appendix B.1 of BMM11, we can use as test statistic

$$\text{TS}_n(\theta) \equiv \int_{\mathcal{H}} \left[\sqrt{n} \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} \bar{m}_n(b, \theta, h) \right]^2 d\Gamma(h),$$

where Γ is a probability measure on \mathcal{H} as explained in Section 3.4 of AS13, and

$$\bar{m}_n(b, \theta, h) \equiv \frac{1}{n} \sum_{i=1}^n m(Y_i, X_i; b, \theta, h).$$

Intuitively, $\text{TS}_n(\theta)$ is built by imposing a penalty for each h such that the maximum of $\mathbb{E}[m(Y_i, X_i; b, \theta, h)]$ across $b \in \mathbb{B}^{|\mathcal{Y}|-1}$ is different from zero. Moreover, given that the support of X_i is finite, the analyst can replace Γ with the uniform probability measure on \mathcal{X} as suggested by Example 5 in Appendix B of AS13. That is,

$$\text{TS}_n(\theta) \equiv \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \left[\sqrt{n} \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} \bar{m}_n(b, \theta, x) \right]^2, \quad (\text{E.4})$$

where

$$\bar{m}_n(b, \theta, x) \equiv \frac{1}{n} \sum_{i=1}^n m(Y_i, X_i; b, \theta) \mathbb{1}\{X_i = x\}.$$

Lastly, we compute the critical value, $\hat{c}_{n,1-\alpha}(\theta)$, by following AS13's bootstrap method consisting of the following steps. First, we draw W_n bootstrap samples using nonparametric i.i.d. bootstrap. Second, for each $w = 1, \dots, W_n$, we compute the recentered test statistic

$$\text{TS}_n^w(\theta) \equiv \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \left[\sqrt{n} \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} (\bar{m}_n^w(b, \theta, x) - \bar{m}_n(b, \theta, x)) \right]^2, \quad (\text{E.5})$$

where $\bar{m}_n^w(b, \theta, x)$ is calculated just as $\bar{m}_n(b, \theta, x)$, but with the bootstrap sample in place of the original sample. Third, $\hat{c}_{n,1-\alpha}(\theta)$ is set equal to the $(1-\alpha)$ quantile of $\{\text{TS}_n^w(\theta)\}_{w=1}^{W_n}$. Once $\text{TS}_n(\theta)$ and $\hat{c}_{n,1-\alpha}(\theta)$ are computed for each $\theta \in \Theta$ (or, in practice, for each θ belonging to a grid), the confidence region, $C_{1-\alpha}$, defined in (E.1) can be constructed.

In Appendix E.1 we provide more details on the computation of (E.4) and (E.5). In particular,

we show that computing (E.4) and (E.5) amounts to solving some quadratically constrained linear programming problems.

E.1 Some computational simplifications

Following MR22, we discuss a way to simplify the computation of the test statistic, $\text{TS}_n(\theta)$, as defined in (E.4). Observe that, for each $x \in \mathcal{X}$ and $b \in \mathbb{B}^{|\mathcal{Y}|-1}$,

$$\bar{m}_n(b, \theta, x) = \mathbb{P}_X(x) b^\top \left(\tilde{\mathbb{P}}_{Y|X}(\cdot|x) - \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right), \quad (\text{E.6})$$

where \mathbb{P}_X is the empirical distribution of X_i , $\tilde{\mathbb{P}}_{Y|X}(\cdot|x) \equiv \begin{pmatrix} \mathbb{P}_{Y|X}(y^1|x) \\ \vdots \\ \mathbb{P}_{Y|X}(y^{|\mathcal{Y}|-1}|x) \end{pmatrix}$, and $\tilde{P}_{Y|X}(\cdot|x) \equiv$

$$\begin{pmatrix} P_{Y|X}(y^1|x) \\ \vdots \\ P_{Y|X}(y^{|\mathcal{Y}|-1}|x) \end{pmatrix}.$$

Therefore, (E.4) is equal to

$$\text{TS}_n(\theta) \equiv \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \left[\sqrt{n} \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} b^\top \left(\mathbb{P}_X(x) \tilde{\mathbb{P}}_{Y|X}(\cdot|x) - \mathbb{P}_X(x) \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right) \right]^2. \quad (\text{E.7})$$

To compute (E.7), the researcher should calculate, for each $x \in \mathcal{X}$,

$$\max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} b^\top \left(\mathbb{P}_X(x) \tilde{\mathbb{P}}_{Y|X}(\cdot|x) - \mathbb{P}_X(x) \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right),$$

which is equivalent to

$$\max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} \min_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} b^\top \left(\mathbb{P}_X(x) \tilde{\mathbb{P}}_{Y|X}(\cdot|x) - \mathbb{P}_X(x) \tilde{P}_{Y|X}(\cdot|x) \right). \quad (\text{E.8})$$

(E.8) is a max-min problem which can be simplified as follows. Note that the inner constrained minimisation problem in (E.8) is linear in $P_{Y|X}(\cdot|x)$. Thus, it can be replaced by its dual, which consists of a linear constrained maximisation problem. Moreover, the outer constrained maxi-

sation problem in (E.8) has a quadratic constraint, $b^\top b \leq 1$. Therefore, (E.8) can be rewritten as a quadratically constrained linear maximisation problem which is a tractable exercise. This is described in detail below.

By (6), (E.8) is equivalent to

$$\begin{aligned}
& \max_{b \in \mathbb{R}^{|\mathcal{Y}|-1}} \min_{P_{Y|X}(\cdot|x), \lambda} && b^\top [\mathbb{P}_X(x) \tilde{\mathbb{P}}_{Y|X}(\cdot|x) - \mathbb{P}_X(x) \tilde{P}_{Y|X}(\cdot|x)], \\
\text{s.t.} \quad & b \in \mathbb{B}^{|\mathcal{Y}|-1}: && b^\top b \leq 1, \\
& \text{Obedience:} && \sum_{k \in \mathcal{K}} \lambda_{k,K}^{y,x} \gamma_{1,k,K}^{y,y',x}(\theta) \geq 0 \quad \forall y \in \mathcal{Y}, \forall y' \in \mathcal{Y} \setminus \{y\}, \\
& \text{Probability requirements:} && \lambda_{k,K}^{y,x} \geq 0 \quad \forall k \in \mathcal{K}, \forall y \in \mathcal{Y}, \quad \sum_{y \in \mathcal{Y}} \lambda_{k,K}^{y,x} = 1 \quad \forall k \in \mathcal{K}, \\
& \text{Choice probability:} && P_{Y|X}(y|x) = \sum_{k \in \mathcal{K}} \lambda_{k,K}^{y,x} \gamma_{2,k,K}(\theta_V) \quad \forall y \in \mathcal{Y}.
\end{aligned} \tag{E.9}$$

We simplify (E.9) by introducing new variables. Let $W_1 \equiv \mathbb{P}_X(x)(\mathbb{P}_{Y|X}(\cdot|x) - P_{Y|X}(\cdot|x))$. Let W be the $(|\mathcal{Y}| + |\mathcal{Y}| \cdot K) \times 1$ vector collecting W_1 and λ . (E.9) can be rewritten as

$$\begin{aligned}
& \max_{b \in \mathbb{R}^{|\mathcal{Y}|-1}} \min_W \left[b^\top \quad 0_{1+|\mathcal{Y}| \cdot K}^\top \right] W, \\
& \text{s.t.} \quad b^\top b \leq 1, \\
& A_{\text{eq}} W = B_{\text{eq}}, \\
& A_{\text{ineq}} W \leq 0_{d_{\text{ineq}}},
\end{aligned} \tag{E.10}$$

where A_{eq} is the matrix of coefficients multiplying W in the equality constraints of (E.9) with d_{eq} rows, B_{eq} is the vector of constants appearing in the equality constraints of (E.9), and A_{ineq} is the matrix of coefficients multiplying W in the inequality constraints of (E.9) with d_{ineq} rows.

Further, the inner constrained minimisation problem in (E.10) is linear. Hence, by strong duality, it can be replaced with its dual. This allows us to solve one unique maximisation problem.

Precisely, the solution of (E.10) is equivalent to the solution of

$$\begin{aligned}
& \max_{\substack{b \in \mathbb{R}^{|\mathcal{Y}|-1} \\ \tau_{\text{eq}} \in \mathbb{R}^{d_{\text{eq}}} \\ \tau_{\text{ineq}} \in \mathbb{R}_+^{d_{\text{ineq}}}}} \begin{bmatrix} -B_{\text{eq}}^\top & 0_{d_{\text{ineq}}}^\top \end{bmatrix} \tau, \\
& \text{s.t. } b^\top b \leq 1, \\
& [A^\top]_{1:|\mathcal{Y}|} \cdot \tau = \begin{pmatrix} -b \\ 0 \end{pmatrix}, \\
& -[A^\top]_{|\mathcal{Y}+1:|\mathcal{Y}+|\mathcal{Y}|\cdot K} \cdot \tau \leq 0_{|\mathcal{Y}|\cdot K},
\end{aligned} \tag{E.11}$$

where τ is the $(d_{\text{eq}} + d_{\text{ineq}}) \times 1$ vector collecting τ_{eq} and τ_{ineq} , A is the $(d_{\text{eq}} + d_{\text{ineq}}) \times (|\mathcal{Y}| + |\mathcal{Y}| \cdot K)$ matrix obtained by stacking one on top of the other the matrices A_{eq} and A_{ineq} , and $[A]_{i:j}$ denotes the sub-matrix of A containing the rows $i, i+1, \dots, j$ of A .

Note that (E.11) is a quadratically constrained linear maximisation problem. In particular, the first constraint in (E.11) is quadratic. The objective function and the remaining constraints in (E.11) are linear. Close derivations are discussed in MR22 for an entry game setting.

We now discuss a way to simplify the computation of bootstrap test statistic, $\text{TS}_n^w(\theta)$, as defined in (E.5). Similarly to (E.6), by rearranging terms it holds that

$$\bar{m}_n^w(b, \theta, x) = \mathbb{P}_X^w(x) b^\top \left(\tilde{\mathbb{P}}_{Y|X}^w(\cdot|x) - \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right),$$

where the superscript “ w ” denotes the bootstrap probabilities. Therefore,

$$\begin{aligned}
& \bar{m}_n^w(b, \theta, x) - \bar{m}_n(b, \theta, x) \\
&= \mathbb{P}_X^w(x) b^\top \left(\tilde{\mathbb{P}}_{Y|X}^w(\cdot|x) - \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right) - P_X^0(x) b^\top \left(\tilde{P}_{Y|X}^0(\cdot|x) - \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right), \\
&= b^\top \left[\mathbb{P}_X^w(x) \tilde{\mathbb{P}}_{Y|X}^w(\cdot|x) - P_X^0(x) \tilde{P}_{Y|X}^0(\cdot|x) - (\mathbb{P}_X^w(x) - P_X^0(x)) \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right].
\end{aligned}$$

To simplify the notation, let us rename

$$A_x^w \equiv \mathbb{P}_X^w(x) \tilde{\mathbb{P}}_{Y|X}^w(\cdot|x) - P_X^0(x) \tilde{P}_{Y|X}^0(\cdot|x),$$

and

$$C_x^w \equiv \mathbb{P}_X^w(x) - P_X^0(x).$$

Therefore, (E.5) is equal to

$$\text{TS}_n^w(\theta) \equiv \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \left[\sqrt{n} \max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} b^\top \left(A_x^w - C_x^w \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right) \right]^2. \quad (\text{E.12})$$

To compute (E.12), the researcher should calculate, for each $x \in \mathcal{X}$,

$$\max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} b^\top \left(A_x^w - C_x^w \max_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} \tilde{P}_{Y|X}(\cdot|x) \right),$$

which is equivalent to

$$\max_{b \in \mathbb{B}^{|\mathcal{Y}|-1}} \min_{P_{Y|X}(\cdot|x) \in \mathcal{Q}(\theta, x)} b^\top \left(A_x^w - C_x^w \tilde{P}_{Y|X}(\cdot|x) \right). \quad (\text{E.13})$$

(E.13) can be rewritten as a quadratically constrained linear maximisation problem as done for (E.8). Once (E.13) is computed for each $x \in \mathcal{X}$, the analyst easily obtains $\text{TS}_n^w(\theta)$.